CRITICAL WETTING FOR A RANDOM LINE IN LONG-RANGE POTENTIAL

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CRITICAL WETTING FOR A RANDOM LINE IN LONG-RANGE POTENTIAL

P. COLLET[(1)], F. DUNLOP[(2)] AND T. HUILLET[(2)]

Abstract. We consider a restricted Solid-on-Solid interface in $\mathbb{Z}_+$, subject to a potential $V(n)$ behaving at infinity like $-\mathrm{w}/n^2$. Whenever there is a wetting transition as $b_0 \equiv \exp V(0)$ is varied, we prove the following results for the density of returns $m(b_0)$ to the origin: If $\mathrm{w} < -3/8$, then $m(b_0)$ has a jump at b_0^c; if $-3/8 < \mathrm{w} < 1/8$, then $m(b_0) \sim (b_0^c - b_0)^{\theta/(1-\theta)}$ where $\theta = 1 - \frac{\sqrt{1-8\mathrm{w}}}{2}$. If $\mathrm{w} > 1/8$, there is no wetting transition.

1. INTRODUCTION and SUMMARY of RESULTS

We consider a restricted Solid-on-Solid (SOS) interface in 1+1 dimension, pinned at the origin, in a potential $V(n)$ characterized by

$$b_n =: e^{V(n)} = 1 - \frac{\mathrm{w}}{n^2} + \mathcal{O}\left(\frac{1}{n^{2+\zeta}}\right),\ \zeta \in (0,1],\ n \in \mathbb{Z}_+ == \{0,1,2,\dots\}.$$

A configuration $(X_i)_{i=0}^N$ with $X_0 = 0$, $X_i \in \mathbb{Z}_+$, $|X_{i+1} - X_i| = 1$, $i = 0, ..., N-1$, has probability

$$\mathbf{P}^{\mathrm{SOS}}(X_1, ..., X_N) \approx \prod_{i=1}^{N} e^{V(X_i)}.$$

Both the free boundary at the endpoint N and the bridge ($X_N = 0$) are considered. Central to this problem is the matrix R obtained while deleting the first row and column of the matrix Q defined by:

$$Q_{p,q} = \begin{cases} \frac{1}{2\sqrt{b_p b_q}}, & \text{if } |p-q| = 1 \\ 0, & \text{otherwise} \end{cases}.$$

The matrix Q (and R) acts on infinite sequences $w = (w_0, w_1, ...)$ (respectively $w = (w_1, w_2, ...)$). We let

$$\mathfrak{w}_1(\rho) = \inf\{w_1 : w > 0 \text{ and } Rw = \rho w - \mathbf{1}_{p=1}\},$$

with $\mathfrak{w}_1(\rho) = \infty$ if the set is empty. If $\mathfrak{w}_1(\rho) < \infty$, we denote by $\mathfrak{w}$ the positive sequence $(\mathfrak{w}_p)_{p\geq 1}$ solution of $R\mathfrak{w} = \rho\mathfrak{w} - \mathbf{1}_{p=1}$, with $\mathfrak{w}_1 = \mathfrak{w}_1(\rho)$. With $\lim_{n\to\infty}(R_{i,j}^{2n})^{1/(2n)} = \rho_*(R) \geq 1$ we define

$$b_0^c = \lim_{\rho \searrow \rho_*(R)} \frac{\mathfrak{w}_1(\rho)}{4b_1\rho}.$$

We show that the SOS model exhibits a (wetting) phase transition as b_0 is varied if and only if R is 1−transient (equivalent to $\mathfrak{w}_1(1) < \infty$ as from Vere-Jones [19]) or equivalently if $b_0^c < \infty$. This can occur only if $\mathrm{w} < 1/8$. If $\mathrm{w} > 1/8$, there is no phase transition. With $\mathfrak{w}_1(\rho(b_0)) = 4\rho(b_0)\, b_0 b_1$ defining $\rho(b_0)$, we show that the

Gibbs potential per site is $-\log \rho\left(b_0\right)$ if $b_0 \leq b_0^c$ and equal to 0 if $b_0 \geq b_0^c$. If $m\left(b_0\right)$ is the density of returns to the origin, we show that

$$\left\{\begin{array}{l} b_0 < b_0^c \Rightarrow m\left(b_0\right) > 0 \\ b_0 > b_0^c \Rightarrow m\left(b_0\right) = 0 \end{array}\right. .$$

Finally, if there is a phase transition, we show that

- if $\mathrm{w} < -3/8$ it is first order: $m\left(b_0\right)$ has a jump at b_0^c,
- if $-3/8 < \mathrm{w} < 1/8$, then $m\left(b_0\right) \sim \left(b_0^c - b_0\right)^{\theta/(1-\theta)}$ as $b_0 \nearrow b_0^c$,

where $\theta = 1 - \frac{\sqrt{1-8\mathrm{w}}}{2}$. This agrees with results by Lipowsky and Nieuwenhuizen [17] who do the computation for a Schrödinger equation of the type

$$\left[-\frac{1}{2}\frac{d^2}{dz^2} + V\left(z\right)\right]\phi(z) = E\phi(z)$$

with $V\left(z\right) = V_0 1_{z\leq z_0} - \mathrm{w}/z^2 1_{z>z_0}$.

The paper is organized as follows:

In Section 2, we develop the relation SOS model versus random walk, allowing to derive an expression for the Gibbs potential.

In Section 3, we focus on the restricted SOS model. We derive the phase diagram in terms of the dominant eigenvalue of the matrix R.

Section 4 is devoted to the study of the density of returns to the origin and corresponding order of the phase transition.

In Section 5, we show that, when the phase transition is continuous, the critical indices are universal in that they only depend on w.

In Section 6, we develop exact results for a particular sequence of (b_n), solved while using Gauss hypergeometric functions.

In Section 7, we develop exact results for a class of sequences (b_n) built from random walks.

Most of the proofs are postponed to the Appendix, Section 8.

2. GIBBS POTENTIAL and RANDOM WALK

2.1. **Background.** We consider a random line or directed polymer $X_0, X_1, \ldots, X_N$ with $X_0 = 0$ and $X_i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with probability distribution

$$\mathbf{P}^{\mathrm{SOS}}(X_1, \ldots, X_N) = Z_N^{-1}\left(\prod_{n=0}^{N-1} e^{-W(X_n, X_{n+1})}\right)\prod_{n=0}^{N} e^{-V(X_n)}, \tag{2.1}$$

where $W(q,p) = W(p,q)$ for all $q, p \in \mathbb{Z}_+$, and Z_N is the partition function normalizing the probability. In SOS model terminology, $V(X_i)$ is the one-body potential.

Here the SOS model represents an interface between two phases at coexistence, interacting with a wall located at $X = 0$. This interaction typically decreases polynomially with the distance to the wall. The zero of energy can be fixed for all such models by requiring

$$\lim_{p\to\infty} \sum_{q\in\mathbb{Z}_+} e^{-W(q,p)} e^{-V(p)/2} = 1, \tag{2.2}$$

and the sum for each $p \in \mathbb{Z}_+$ is assumed to converge. We will be mostly interested in knowing whether the line (interface) stays in the vicinity of the wall (partial wetting) or escapes to infinity (complete wetting).

In the sequel we will use Landau's notation $\sim$, namely for two sequences (a_n) and (b_n), $(a_n) \sim (b_n)$ means

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Similarly, $a_n \approx b_n$ is when the limit is any non-zero constant instead of 1.

2.2. **Computation of the Gibbs potential.** The Gibbs potential is defined by

$$\Phi\left((b_n)\right) = \lim_{N\to\infty} -\frac{1}{N}\log Z_N. \tag{2.3}$$

In order to represent (2.1) as the probability of a random walk trajectory, possibly weighted at its end-point X_N, let us assume for some $\rho > 0$ the existence of a solution U depending upon ρ to

$$\sum_{q\in\mathbb{Z}_+} e^{-U(q)/2} e^{-W(q,p)-V(q)/2-V(p)/2} = \rho\, e^{-U(p)/2},\ p \geq 0 \tag{2.4}$$

and define a random walk starting at $X_0 = 0$ with values in $\mathbb{Z}_+$ by the transition probabilities

$$\mathbf{P}^{\mathrm{RW}}\left(X_{n+1} = p \mid X_n = q\right) = \rho^{-1} e^{-W(q,p)-V(q)/2-V(p)/2-U(p)/2+U(q)/2},\ q,p \geq 0. \tag{2.5}$$

Note that (2.4) implies that (2.5) is properly normalized. Moreover (2.5) implies that the walk obeys the detailed balance condition with respect to the unnormalized measure $\exp(-U(q))$ over $\mathbb{Z}_+$. Also (2.5) gives

$$e^{-W(q,p)-V(p)/2-V(q)/2} = \rho \cdot \left(\mathbf{P}^{\mathrm{RW}}(p,q)\mathbf{P}^{\mathrm{RW}}(q,p)\right)^{1/2}. \tag{2.6}$$

The SOS model and the random walk started at $X_0 = 0$ are related by

$$\mathbf{P}^{\mathrm{SOS}}(X_1,\ldots,X_N) = Z_N^{-1}\rho^N \mathbf{P}^{\mathrm{RW}}(X_1,\ldots,X_N)\, e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}, \tag{2.7}$$

and their marginal

$$\mathbf{P}^{\mathrm{SOS}}(X_N) = Z_N^{-1}\rho^N \mathbf{P}^{\mathrm{RW}}(X_N)\, e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}. \tag{2.8}$$

$\mathbf{P}^{\mathrm{SOS}}(X_N)$ and $\mathbf{P}^{\mathrm{RW}}(X_N)$ may differ strongly due to the factor $e^{\frac{1}{2}U(X_N)}$, but conditioned on the value of X_N, the distribution of $X_1,\ldots,X_{N-1}$ is the same for SOS and for a corresponding random walk. This correspondence between random walk and random line was developed in [16] and [5].

2.3. **Bridge.** For the bridge $(X_{2N} = 0)$ the partition function is given by

$$\begin{aligned} Z_{2N} &= \sum_{X_1,\ldots,X_{2N-1}} \left(\prod_{n=0}^{2N-2} e^{-W(X_n,X_{n+1})}\right)\left(\prod_{n=0}^{2N-1} e^{-V(X_n)}\right) e^{-W(X_{2N-1},0)} e^{-V(0)} \\ &= \rho^{2N} e^{-V(0)} \mathbf{P}^{\mathrm{RW}}(X_{2N} = 0). \end{aligned} \tag{2.9}$$

Hence if

$$\lim_{N\to\infty} \frac{1}{2N}\log \mathbf{P}^{\mathrm{RW}}(X_{2N} = 0) = 0,$$

the Gibbs potential is equal to $-\log\rho$.

Remark 1. *If the walk has a normalizable invariant measure the above condition is satisfied. If the walk has a non-normalizable invariant measure, it may happen that* $\mathbf{P}^{\mathrm{RW}}(X_{2N}=0)$ *decay exponentially fast with* N*. In that case the Gibbs potential is not* $-\log\rho$*.*

2.4. **Free boundary condition.** Summing over X_N in (2.8) we get

$$Z_N=\rho^N\sum_{X_N}\mathbf{P}^{\mathrm{RW}}(X_N)\,e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}. \tag{2.10}$$

Here the situation is more delicate because the function $e^{\frac{1}{2}U(X_N)}$ may diverge. If

$$\lim_{N\to\infty}\frac{1}{N}\log\sum_{X_N}\mathbf{P}^{\mathrm{RW}}(X_N)\,e^{\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}=0,$$

the Gibbs potential is $-\log\rho$. If $|V|$ is bounded we only have to look at the behavior for large N of

$$\sum_{X_N}\mathbf{P}^{\mathrm{RW}}(X_N)\,e^{\frac{1}{2}U(X_N)}.$$

By detailed balance, for every p, we have

$$\mathbf{P}^{\mathrm{RW}}(X_N=p\,|\,X_0=0)\,e^{U(p)}\,e^{-\frac{1}{2}U(p)}=e^{U(0)}\,\mathbf{P}^{\mathrm{RW}}(X_N=0\,|\,X_0=p)\,e^{-\frac{1}{2}U(p)}$$

and the bounds (see also lemma 25)

$$\begin{aligned} e^{U(0)/2}\mathbf{P}^{\mathrm{RW}}(X_N=0\mid X_0=0)&\le e^{U(0)}\sum_p\mathbf{P}^{\mathrm{RW}}(X_N=0\,|\,X_0=p)\,e^{-\frac{1}{2}U(p)}\\ &\le e^{U(0)}\,(N+1)\cdot\sup_{0\le p\le N}e^{-\frac{1}{2}U(p)}.\end{aligned} \tag{2.11}$$

Therefore, if $\left(e^{-\frac{1}{2}U(p)}\right)$ is a bounded sequence and

$$\lim_{N\to\infty}\frac{1}{2N}\log\mathbf{P}^{\mathrm{RW}}(X_{2N}=0)=0, \tag{2.12}$$

the Gibbs potential is equal to $-\log\rho$. This applies to random walks with period one (irreducible) or two.

3. The CASE $X_{n+1}-X_n=\pm1$

For $q-p=\pm1$, the normalization (2.2) is satisfied with $W(q,p)=\log 2$ and $V(q)\to0$ as $q\to\infty$. Therefore

$$W(q,p)+\frac{1}{2}V(q)+\frac{1}{2}V(p)=\begin{cases}\ln 2+\frac{1}{2}V(q)+\frac{1}{2}V(p) & \text{if } p-q=\pm1\\ +\infty & \text{otherwise}\end{cases}. \tag{3.1}$$

Letting

$$b_p=e^{V(p)}\text{ and }v_p=e^{-U(p)/2},$$

equation (2.4) reads

$$Qv=\rho v\text{ with}$$

$$Q_{p,q}=\begin{cases}\frac{1}{2\sqrt{b_pb_q}},\ \text{if } |p-q|=1\\ 0,\ \text{otherwise}\end{cases}$$

so that

$$(Qv)_p=\begin{cases}\frac{1}{2\sqrt{b_0b_1}}v_1,\ \text{for } p=0\\ \frac{1}{2\sqrt{b_pb_{p+1}}}v_{p+1}+\frac{1}{2\sqrt{b_pb_{p-1}}}v_{p-1},\ \text{for } p>0\end{cases}.$$

We will sometimes write Q_{b_0} instead of Q in order to emphasize the dependence in b_0, our main parameter below.

In general there is a continuum of values of ρ such that there exists a positive solution to $Qv = \rho v$, but there is only one Gibbs potential. In the case of the free boundary condition, the other solutions with a $\rho \neq e^{-\Phi}$ leave a non trivial boundary term in the relation (2.10). This gives an exponential correction leading finally to the right Gibbs potential. Assume we have a positive solution of $Qv = \rho v$. Then

$$\frac{v_2}{2\sqrt{b_2 b_1}} + \frac{v_0}{2\sqrt{b_0 b_1}} = \rho v_1$$

can be rewritten as

$$\frac{v_2}{2\sqrt{b_2 b_1}} = \rho v_1 - \frac{v_0}{2\sqrt{b_0 b_1}},$$

which means that (w_p) defined for $p \geq 1$ by

$$w_p = \frac{2 v_p \sqrt{b_0 b_1}}{v_0}$$

is a positive solution of

$$Rw = \rho w - \mathbf{1}_{p=1}, \tag{3.2}$$

where R denotes the matrix Q without its first row and first column,

$$(Rv)_p = \begin{cases} \frac{1}{2\sqrt{b_1 b_2}} v_2, \text{ for } p = 1 \\ \frac{1}{2\sqrt{b_p b_{p+1}}} v_{p+1} + \frac{1}{2\sqrt{b_p b_{p-1}}} v_{p-1}, \text{ for } p > 1 \end{cases}.$$

Note that R is independent of b_0.

In the terminology of [19], the matrix R must be $\rho-$transient. Indeed, according to Corollary 4. Criterion II in [19], the matrix R is $\rho-$transient if and only if equation (3.2) has a positive solution. Else, R is $\rho-$recurrent.

For convenience we will use $\{1, 2, \ldots\}$ for the indices of R. We also have

$$\frac{v_1}{2\sqrt{b_0 b_1}} = \rho v_0,$$

hence

$$w_p = \frac{4 \rho v_p b_0 b_1}{v_1}$$

and in particular

$$w_1 = 4 \rho b_0 b_1.$$

Let

$$\mathfrak{w}_1(\rho) = \inf\left\{w_1 : w > 0 \text{ and } Rw = \rho w - \mathbf{1}_{p=1}\right\},$$

with $\mathfrak{w}_1(\rho) = \infty$ if the condition leads to an empty set. Then

$$4 \rho b_0 b_1 \geq \mathfrak{w}_1(\rho)$$

or in other words

$$\frac{\mathfrak{w}_1(\rho)}{4 \rho b_1} \leq b_0.$$

This condition is thus necessary and sufficient for the equation $Qv = \rho v$ to have a positive solution.

As will be seen in detail below, many properties of the model depend on the function $\mathfrak{w}_1(\rho)$. We now recall some results by Vere-Jones (see [19]) adapted to our setting.

Theorem 1. *(i) The limit*

$$\rho_* = \lim_{n\to\infty} (R^{2n}_{i,j})^{1/(2n)}$$

exists and is independent of (i,j) for all $i-j$ even.

(ii)

$$\rho_* = \inf\{\rho :\ \exists w \geq 0,\ Rw = \rho w - \mathbf{1}_{n=0}\}.$$

(iii) For $\rho < \rho_*$ the equation $Rw = \rho w - \mathbf{1}_{n=0}$ has no positive solution.

Proof:
(i) follows from Theorem A in [19].
(ii) follows from Corollary 4 in [19].
(iii) follows from Corollary 1 in [19]. □

The latter theorem holds under more general conditions. In the case of our Jacobi matrices Q or R we can get the following more precise results which we have not found in the literature.

Theorem 2. (i) $\liminf_p \frac{1}{\sqrt{b_p b_{p+1}}} \leq \rho_* \leq \sup_{p\geq 1} \frac{1}{\sqrt{b_p b_{p+1}}}$.

(ii) If $\lim_{n\to\infty} b_n = 1$, then $1 \leq \rho_* < \infty$.

(iii) $\forall \rho > 0$, the equation $Qv = \rho v$ has a unique solution modulo a constant factor.

(iv) If there is a positive solution to (3.2), then the equation $Rv = \rho v$ has a positive solution.

Proof: The proof is given in Appendix $A.1$. Note that the v in (iii) is not necessarily positive.

From now on we will assume that

$$\sum_{n=1}^{\infty} |1 - b_n| < \infty, \tag{3.3}$$

which implies of course $\lim_{n\to\infty} b_n = 1$. We will denote by $\mathfrak{w}$ the sequence $(\mathfrak{w}_p)_{p\geq 1}$ solution of

$$R\mathfrak{w} = \rho\mathfrak{w} - \mathbf{1}_{p=1}, \tag{3.4}$$

with $\mathfrak{w}_1 = \mathfrak{w}_1(\rho)$. Note that by continuity, $(\mathfrak{w})$ is a non-negative sequence and from the recursion, in fact positive.

Lemma 3. *We have*

(i) The function $\mathfrak{w}_1(\rho)$ is decreasing and continuous in ρ for $\rho \in (\rho_*(R), \infty)$.

(ii) The function $\rho^{-1}\mathfrak{w}_1(\rho)$ is decreasing and continuous in ρ for $\rho \in (\rho_*(R), \infty)$.

(iii)

$$\lim_{\rho\to\infty} \frac{\mathfrak{w}_1(\rho)}{\rho} = 0.$$

(iv) If $\rho_*(R) > 1$,

$$\lim_{\rho\searrow\rho_*(R)} \mathfrak{w}_1(\rho) = \infty,\ \textit{hence}\ \lim_{\rho\searrow\rho_*(R)} \frac{\mathfrak{w}_1(\rho)}{\rho} = \infty.$$

(v) If $\rho_*(R) = 1$ and $\lim_{\rho\searrow 1} \mathfrak{w}_1(\rho) < \infty$, then

$$\lim_{\rho\searrow 1} \mathfrak{w}_1(\rho) = \mathfrak{w}_1(1).$$

Proof: The proof is given in Appendix $A.2$.

As will be seen below, the existence or not of a phase transition is related to the property that $\rho_*(R) = 1$ and R is one-transient. This corresponds to the situation $\rho_*(R) = 1$ and $\lim_{\rho \searrow 1} \mathfrak{w}_1(\rho) < \infty$ (see Lemma 3). We have not found a general criterion to decide if this property is true or not for a general sequence (b_n).

Besides the explicit example given later on, we can deal with several cases.

Proposition 4. *If $b_n \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} b_n = 1$, then $\rho_*(R) = 1$ and R is one-transient.*

Proof: The proof is given in Appendix $A.3$.

We will be mostly interested later on by sequences (b_n) such that for $n \geq 1$

$$b_n = 1 - \frac{\mathrm{w}}{n^2} + \mathcal{O}\left(\frac{1}{n^{2+\zeta}}\right),\ \zeta > 0. \tag{3.5}$$

Proposition 5. *Assume the sequence (b_n) satisfies (3.5). Then*

(i) For $\mathrm{w} > 1/8$ the equation $Rw = w - \mathbf{1}_{p=1}$ has no positive solution.

(ii) For any $\mathrm{w} < 1/8$, there exists a positive sequence (b_n) satisfying (3.5) such that the equation $Rw = w - \mathbf{1}_{p=1}$ has no positive solution.

(iii) For any $\mathrm{w} < 1/8$, there exists a positive sequence (b_n) satisfying (3.5) such that the equation $Rw = w - \mathbf{1}_{p=1}$ has a positive solution.

(iv) For the sequence $b_n = 1 - \frac{\mathrm{w}}{n^2}$ for all $n \geq 1$, there exists $0 < \mathrm{w}_c \leq 1/8$ such that for any $\mathrm{w} < \mathrm{w}_c$, the equation $Rw = w - \mathbf{1}_{p=1}$ has a positive solution.

Proof: The proof is given in Appendix $A.4$.

We have performed numerical simulations suggesting that in case (iv), $\mathrm{w}_c = 1/8$.

3.1. **Gibbs potential revisited.** We define

$$b_0^c = \lim_{\rho \searrow \rho_*(R)} \frac{\mathfrak{w}_1(\rho)}{4 b_1 \rho}.$$

Note that $b_0^c > 0$ may be infinite, and by Lemma 3, $b_0^c < \infty$ implies $\rho_*(R) = 1$. In this case $\mathfrak{w}_1(1) < \infty$ (R is $1-$transient).

Lemma 6. *Assume $\lim_{n\to\infty} b_n = 1$. Consider both the free and zero boundary condition (bridge).*

(i) If $b_0 < b_0^c$, there is a unique $\rho(b_0)$ (which is larger than one) such that

$$\frac{\mathfrak{w}_1(\rho(b_0))}{4\rho(b_0) b_1} = b_0,$$

and $\rho(b_0) = \rho_*(Q_{b_0})$ and the Gibbs potential coincides with $-\log \rho(b_0)$.

(ii) Assume $b_0^c < \infty$ and $b_0 > b_0^c$, then the Gibbs potential is equal to zero.

Proof: The proof is given in Appendix $A.5$.

When $b_0^c < \infty$, this result is a hint for the existence of a phase transition.

4. DENSITY of RETURNS to the ORIGIN and PHASE TRANSITION

Recall (see 2.5) that if the equation $Qv = \rho v$ has a positive solution, the walk on $\mathbb{Z}_+$ reflected at zero given by for $n \geq 1$

$$p_n = \frac{1}{\rho\sqrt{b_n b_{n+1}}} \frac{v_{n+1}}{v_n},$$

(and $p_0 = 1$) has a positive invariant measure (π_n) (not necessarily normalizable) given by

$$\pi_n = v_n^2.$$

Recall also that v is unique up to a positive factor. When $v \in \ell^2(\mathbb{Z}_+)$, we will denote by (ν_n) the invariant probability measure

$$\nu_n = \frac{\pi_n}{\sum_{j=0}^{\infty} \pi_j} = \frac{v_n^2}{\sum_{j=0}^{\infty} v_j^2}.$$

In the sequel, for a given $b_0 < b_0^c$ (and for $b_0 = b_0^c$ if $b_0^c < \infty$) we will take $\rho = \rho(b_0)$.

Proposition 7. *Assume $b_0 < b_0^c$ in which case the random walk is positive recurrent. Then*

(i) the following limits (density of returns to the origin) exist

$$\lim_{N\to\infty} \frac{1}{2N-1} \sum_{X_1,\ldots,X_{2N-1}} \mathbf{P}^{SOS}(X_1,\ldots,X_{2N-1} \mid X_{2N}=0) \sum_{l=1}^{2N-1} \mathbf{1}_{X_l=0}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{X_1,\ldots,X_N} \mathbf{P}^{SOS}(X_1,\ldots,X_N) \sum_{l=1}^{N} \mathbf{1}_{X_l=0}.$$

(ii) Moreover, these two limits are equal to

$$m(b_0) = \nu_0 = \frac{v_0^2}{\sum_{p=0}^{\infty} v_p^2} = \frac{1}{1 + (4b_0\, b_1)^{-1} \sum_{p=1}^{\infty} \mathfrak{w}_p(\rho(b_0))^2}. \tag{4.1}$$

(iii) The function $m(b_0)$ is non-increasing.

(iv) The Gibbs potential $\Phi((b_n))$ has a partial derivative with respect to b_0 equal to $m(b_0)/b_0$.

Note that m (the density of returns to the origin) is equal to zero if the denominator diverges, namely if (π_n) is not normalizable.

Proof: The proof is given in Appendix A.6.

Theorem 8. *Assume* (3.5). *Then*

(i) For any $b_0 < b_0^c$, $m(b_0) > 0$.

(ii) Assume $b_0^c < \infty$ and $b_0 > b_0^c$, then $m(b_0) = 0$.

(iii) Assume $b_0^c < \infty$, if $-3/8 \le \mathrm{w} < 1/8$ then

$$\lim_{b_0 \nearrow b_0^c} m(b_0) = 0.$$

(iv) Assume $b_0^c < \infty$, if $\mathrm{w} < -3/8$ then

$$\lim_{b_0 \nearrow b_0^c} m(b_0) > 0.$$

Proof: The proof is given in Appendix A.8.

We note that, whenever $\mathrm{w} < -3/8$ and $b_0^c < \infty$, the density of returns $m(b_0)$ has a jump at b_0^c.

5. UNIVERSALITY of CRITICAL INDICES

We show the following

Theorem 9. *If the sequence (b_n) satisfies*

$$b_n = 1 - \frac{\mathrm{w}}{n^2} + \mathcal{O}\left(n^{-2-\zeta}\right) \tag{5.1}$$

for some $1 \geq \zeta > 0$ with $-3/8 < \mathrm{w} < 1/8$, and if R is 1-transient, then the sequence $(\mathfrak{w}_p(1+\epsilon))$ $(\epsilon > 0)$ satisfies

$$0 < \liminf_{\epsilon \nearrow 0} \epsilon^{\theta} \sum_{p=1}^{\infty} \mathfrak{w}_p(1+\epsilon)^2 \leq \limsup_{\epsilon \nearrow 0} \epsilon^{\theta} \sum_{p=1}^{\infty} \mathfrak{w}_p(1+\epsilon)^2 < \infty \tag{5.2}$$

where $\theta = 1 - \frac{\sqrt{1-8\mathrm{w}}}{2}$.

Remark: Observe that for $-3/8 \leq \mathrm{w} < 1/8$,

$$0 \leq \frac{\theta}{1-\theta} < \infty.$$

(the transformation $\mathrm{w} \to \theta(\mathrm{w})/(1-\theta(\mathrm{w})$ maps bijectively the interval $[-3/8, 1/8[$ on $\mathbb{R}^+$). The condition $\zeta \leq 1$ is of course non restrictive but will be convenient in the estimates later on.

We will use the argument developed in Appendix $A.10$ to determine the value of the critical index, but now in the general case. Recall indeed that (see Proposition 7.(iv))

$$m(b_0) = -\frac{b_0}{\rho_0(b_0)} \partial_{b_0} \rho_0(b_0).$$

From (5.2) and Proposition 7.(ii), if $\rho(b_0) = 1 + \epsilon(b_0)$ we have

$$m(b_0) \approx \epsilon(b_0)^{\theta}.$$

Therefore

$$\partial_{b_0} \rho_0(b_0) = \partial_{b_0} \epsilon \approx -\frac{\rho_0(b_0)}{b_0} \epsilon(b_0)^{\theta}.$$

This implies

$$b_0^c - b_0 \approx \epsilon^{1-\theta}$$

and therefore we obtain the

Corollary 10. *Under the hypothesis of Theorem 9, the density of returns to the origin obeys*

$$m(b_0) \approx (b_0^c - b_0)^{\theta/(1-\theta)} \quad \textit{as } b_0 \nearrow b_0^c.$$

Remark: In (8.7), $s = \alpha_+$ is the "other" solution of

$$s^2 - s + 2\mathrm{w} = 0, \tag{5.3}$$

namely

$$s = 1 - \alpha_-,$$

and we get

$$\frac{3/2 - s}{s - 1/2} = \frac{\alpha_- + 1/2}{1/2 - \alpha_-} = \frac{\theta}{1-\theta},$$

as expected from the results for the hypergeometric model developed in the next Section.

The proofs for critical indices are postponed to Appendix $A.14$.

6. The HYPERGEOMETRIC MODEL: a SOLVABLE CASE

Up to now, in our discussion, we presented rather general results. For a particular choice of the sequence $(b_n)_{n\geq 1}$, one can derive more explicit expressions.

6.1. The sequences (b_n) and $(\mathfrak{w}_p)$. Let $s \geq 1/2$ and $a > 3/4$ (other parameter ranges are possible). For $n \geq 1$ define

$$b_n = \frac{(s+n-2+2a)\,\Gamma(a+n/2-1/2)\,\Gamma(s+a+n/2-1)}{2\,\Gamma(a+n/2)\,\Gamma(s+a+n/2-1/2)}.$$

Let

$$V(n) = \log b_n,$$

then

$$\lim_{n\to\infty} V(n) = 0$$

and

$$b_n = 1 - \frac{\mathrm{w}}{n^2} + \mathcal{O}(n^{-3})$$

with

$$\mathrm{w} = \frac{s-s^2}{2}.$$

Note that $\mathrm{w} \leq 1/8$, and the half line $s \in [1/2,\infty)$ maps to the half line $\mathrm{w} \in (-\infty, 1/8]$.

Theorem 11. *It holds that*

(i)

$$\mathfrak{w}_p\,(\rho) = 2(2\rho)^{-p}\frac{F(a+(p-1)/2, a+p/2; 2a+p+s-1; \rho^{-2})}{F(a-1/2, a; 2a+s-1; \rho^{-2})}\times$$

$$\sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)\Gamma(2a+p-1)\Gamma(2s+2a+p-2)}{\Gamma(s+p-2+2a)\Gamma(s+p-1+2a)\Gamma(2a)\Gamma(2s+2a-1)}}.$$

(ii) $\rho_*(R) = 1$

(iii) We have

$$b_0^c = \frac{\mathfrak{w}_1(1)}{4b_1} = \frac{1}{2}\frac{\Gamma(a+s-1)\Gamma(a+1/2)}{\Gamma(a)\Gamma(s+a-1/2)},$$

and for $0 < b_0 \leq b_0^c$, $\rho(b_0)$ is the unique solution larger than one of the implicit equation

$$b_0 = \frac{\mathfrak{w}_1(\rho(b_0))}{4\rho(b_0)\,b_1} = \frac{1}{4\rho(b_0)^2 b_1}\frac{F(a,a+1/2;2a+s;\rho^{-2}(b_0))}{F(a-1/2,a;2a+s-1;\rho^{-2}(b_0))}.$$

(iv)

$$\mathfrak{w}_1(1) \sim p^{1-s}.$$

Here $F =_2 F_1$ the hypergeometric function. The proof is given in Appendix $A.9$.

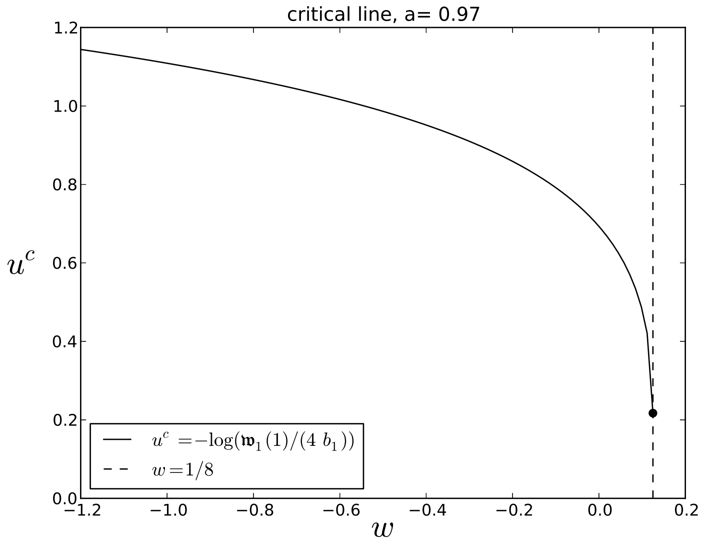

FIGURE 1. The critical line $u^c = -\log(\frac{\mathfrak{w}_1(1)}{4\,b_1})$.

6.2. **Thermodynamics of the hypergeometric model.** One can think of w as some normalized inverse temperature and $u := -\log b_0 = -V(0)$ (or better u/w) as pressure. Because u and m are intensive variables, $-\log\rho$ is a Gibbs potential.

Proposition 12. (i) *For any* $0 < b_0 < b_0^c$*, with* $\rho = \rho(b_0)$

(6.1)
$$m = \frac{1}{2 + \frac{2}{\rho^2}\frac{F(a+1,a+3/2;2a+s+1;\rho^{-2})}{F(a,a+1/2;2a+s;\rho^{-2})}\frac{a(a+1/2)}{2a+s} - \frac{2}{\rho^2}\frac{F(a+1/2,a+1;2a+s;\rho^{-2})}{F(a-1/2,a;2a+s-1;\rho^{-2})}\frac{a(a-1/2)}{2a+s-1}}.$$

(ii) *For* $1/2 < s < 3/2$ *(*$-3/8 < \mathrm{w} < 1/8$*)*
$$\lim_{b_0 \nearrow b_0^c} m(b_0) = 0,$$
and
$$\lim_{b_0 \nearrow b_0^c} \frac{m(b_0, s)}{(b_0^c - b_0)^{(3/2-s)/(s-1/2)}}$$
exists, is finite and non zero. The critical index $(3/2-s)/(s-1/2)$ *can be expressed in terms of* w *using the relation* $\mathrm{w} = (s - s^2)/2$*.*

(iii) *For* $s > 3/2$ *(*$\mathrm{w} < -3/8$*)*
$$\lim_{b_0 \nearrow b_0^c} m(b_0, s) = \frac{1}{2 + \frac{2a}{s-3/2}}.$$

For the proof, see Appendix $A.10$ 8.

In figure 1, with $a = 0.97$, we plot the critical line
$$u^c := -\log(b_0^c) = -\log\left(\frac{\mathfrak{w}_1(1)}{4b_1}\right)$$
as a function of w.

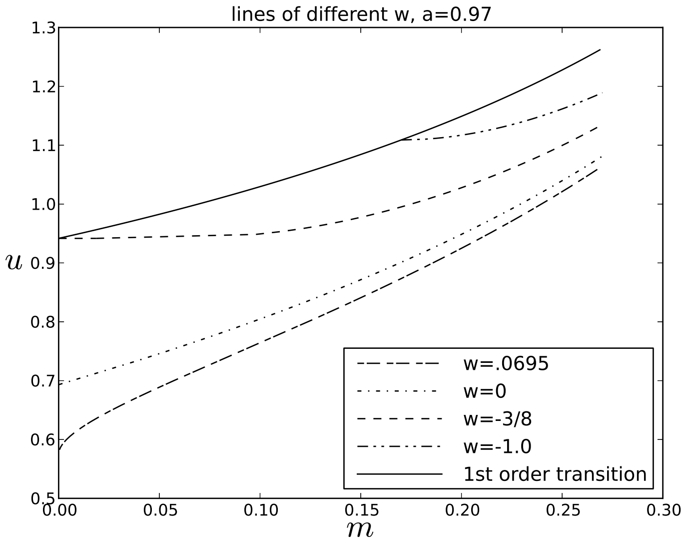

FIGURE 2. The thermodynamic diagram in the plane (m,u).

In figure 2, we plot the thermodynamic diagram in the plane (m,u), with lines corresponding to various values of w. The red line corresponds to the first order phase transition, namely the inverse function of $u \to m(\exp(-u), \mathrm{w}(u))$ with $\mathrm{w}(u)$ such that $\rho(\exp(-u), \mathrm{w}(u)) = 1$.

6.3. **Particular values of** s**.** The formulas simplify for s integer, we only treat the cases $s = 1$ and $s = 2$.
$s = 1$ ($\mathrm{w} = 0$). In that case it is easy to verify that $b_n = 1$ for any $n \geq 1$. Also $\mathfrak{w}_n = 2e^{-nv}$ for $n \geq 1$ with

$$\cosh(v) = \rho.$$

The equation for ρ is

$$\mathfrak{w}_1 = 2e^{-v} = 4\rho b_0$$

hence $b_0^c = 1/2$. For $0 < b_0 < b_0^c$ we have

$$\rho(b_0) = -\log 2 - \frac{1}{2}\left(\log b_0 + \log(1 - b_0)\right),$$

and

$$m(b_0) = \frac{1/2 - b_0}{1 - b_0}.$$

See Appendix $A.11$ for the details of the computations. Note that in accordance with Proposition 12,

$$\lim_{b_0 \nearrow 1/2} \frac{m(b_0)}{1/2 - b_0} = 2.$$

6.3.1. $s = 2$ *(*$\mathrm{w} = -1$*).* In this case we have

$$b_p = \frac{(2a+p)^2}{(2a+p)^2 - 1},$$

$$\mathfrak{w}_p = 2\rho^{-p}(1+\sqrt{1-\rho^{-2}})^{-p}\frac{(a+p/2-1/2)\sqrt{1-\rho^{-2}} - a - p/2 + 1/2 + \rho^{-2}(a+p/2)}{(a+p/2+1/2)}\times$$

$$\frac{a+1/2}{(a-1/2)\sqrt{1-\rho^{-2}} - a + 1/2 + a\,\rho^{-2}}\sqrt{\frac{a\,(2a+p+1)}{(2a+p-1)\,(a+1)}}$$

and

$$b_0^c = \frac{a}{2a+1}.$$

For $0 < b_0 < b_0^c$ we have

$$\rho(b_0) = -\frac{1}{2}\log\left(1 - \frac{\left(\sqrt{{b_0^c}^2\,(b_0^c - b_0)^2 + (b_0^c - b_0)\,(b_0^c - 2{b_0^c}^2)} + b_0^c\,(b_0^c - b_0)\right)^2}{{b_0^c}^2}\right)$$

$$m(b_0) = -\frac{\text{num}}{\text{den}},$$

where

$$\begin{aligned}\text{num} \quad &= \quad \sqrt{b_0^c}\left(4b_0{b_0^c}^3 - \left(8b_0^2 + 6b_0\right){b_0^c}^2 + \left(4b_0^3 + 6b_0^2 + 3b_0\right)b_0^c - 3b_0^2\right) + \\ &\qquad \sqrt{b_0^c - b_0}\sqrt{{b_0^c}^2 - (b_0+2)\,b_0^c + 1}\left(4b_0{b_0^c}^2 - \left(4b_0^2 + 2b_0\right)b_0^c + b_0\right),\end{aligned}$$

$$\begin{aligned}\text{den} \quad = \quad &\sqrt{b_0^c}\left(4{b_0^c}^4 - (12b_0+8)\,{b_0^c}^3 + \left(12b_0^2 + 16b_0 + 4\right){b_0^c}^2 - \left(4b_0^3 + 8b_0^2 + 8b_0\right)b_0^c + 4b_0^2\right) + \\ &\sqrt{b_0^c - b_0}\sqrt{{b_0^c}^2 - (b_0+2)\,b_0^c + 1}\left(4{b_0^c}^3 - (8b_0+4)\,{b_0^c}^2 + \left(4b_0^2 + 4b_0\right)b_0^c - 2b_0\right)\end{aligned}$$

and

$$\lim_{b_0 \nearrow b_0^c} m(b_0) = \frac{1}{4a+2}.$$

See Appendix $A.12$ for the details of the computations.

7. From RANDOM WALK to SOS MODEL

In this Section, we supply a class of interesting random walks on the integers (reflected at the origin) akin to the discrete Bessel model. From the probabilities (p_n, q_n) to move up (and down) by one unit given the walker is in state n with $p_n + q_n = 1$, $n \geq 1$, the sequence (b_n) of corresponding SOS model is given by the recurrence

$$b_n b_{n+1} = \frac{1}{4p_n q_{n+1}},\; n \geq 0,$$

allowing to compute $(b_n)_{n\geq 1}$ as a function of b_0. We shall assume $p_n \to 1/2$ as $n \to \infty$ (the random walk has zero drift at infinity) and furthermore $p_n \sim \frac{1}{2}\left(1 + \frac{\lambda}{n} + \frac{A}{n^2}\right)$ for some λ as $n \to \infty$, compatible with $b_n = 1 - \frac{\mathrm{w}}{n^2} + \mathcal{O}\left(\frac{1}{n^{2+\zeta}}\right)$ for $\mathrm{w} = \frac{1}{2}\lambda\,(1-\lambda)$.

Letting indeed $B_{k+1}=b_{2k+1}$ and $C_k=b_{2k}$, we find

$$\begin{aligned} B_{k+1} &= B_k \frac{p_{2k-1}q_{2k}}{p_{2k}q_{2k+1}} =: B_k U_{k+1},\ B_0=\frac{1}{4b_0q_1}=b_1 \\ C_{k+1} &= C_k \frac{p_{2k}q_{2k+1}}{p_{2k+1}q_{2k+2}} =: C_k V_{k+1},\ C_0=b_0 \end{aligned}$$

where $B_kC_k=b_{2k}b_{2k+1}=\frac{1}{4p_{2k}p_{2k+1}}\to 1$ as $k\to\infty$. Thus,

$$\begin{aligned} B_k &= B_0\prod_{l=1}^{k}U_l \underset{k\to\infty}{\to} B_0u=1 \\ C_k &= C_0\prod_{l=1}^{k}V_l \underset{k\to\infty}{\to} C_0v=1 \end{aligned}$$

where $v=\prod_{l=1}^{\infty}\frac{p_{2l-2}q_{2l-1}}{p_{2l-1}q_{2l}}=1/b_0$ and $u=4b_0q_1$. This shows that there is a unique value of b_0 for which $b_n\to 1$ as $n\to\infty$.

More generally, for $\rho>1$, we can build the sequence (b_n) of an SOS model corresponding to a random walk while using the recurrence

$$b_nb_{n+1}=\frac{1}{4\rho^2p_nq_{n+1}},\ n\geq 0.$$

We would conclude proceeding similarly that there is a unique value of $b_0=b_0(\rho)$ for which $b_n=b_n(\rho)\to 1$ as $n\to\infty$. The latter recurrence can be represented by the matrix

$$Q=\rho P_S$$

where, with P the transition matrix of the (reversible) random walk and π its speed measure solution to $\pi=\pi P$, $P_S=D_\pi^{1/2}PD_\pi^{-1/2}$ is the symmetrized version of P. We used $D_\pi=$diag$(\pi_0,\pi_1,...)$. The matrix Q is the one defined in Section 3 and $Qv=\rho v$ with $v_n=\sqrt{\pi_n}>0$, $n\geq 0$. The speed measure formula for (π_k), for $k>0$, is

$$\pi_k=\frac{\pi_0}{q_k}\prod_{j=1}^{k-1}\frac{p_j}{q_j}=\frac{p_{k-1}}{q_k}\pi_{k-1}. \tag{7.1}$$

We now come to our special class of random walks.

7.1. **Bessel random walks.** Let $x_0,d>0$ be parameters. With $R_n=n+x_0$, $n\geq 0$ integer, the radii of balls of dimension d with area and volume

$$A(R_n)=\frac{2\pi^{d/2}}{\Gamma(d/2)}R_n^{d-1}\text{ and }V(R_n)=\frac{\pi^{d/2}}{\Gamma(d/2+1)}R_n^d,$$

we are interested in a random walk in concentric nested balls of radii R_n. Although $V(R_n)>V(R_{n-1})$, always when $d>0$, we note that if $d>1$, $A(R_n)>A(R_{n-1})$, while if $d<1$, $A(R_n)<A(R_{n-1})$. The domain confined between ball number n and ball number $n-1$, $n\geq 1$, is an annulus with volume $V(R_n)-V(R_{n-1})$; $V(R_0)$ is the volume of the central ball.

Negative dimensions can be meaningful as well: indeed, the Euler gamma function $\Gamma(\alpha)$ is positive when α lies in the intervals $\alpha\in(-2k,-2k+1)$, $k\geq 0$. To have both $A(R_n),V(R_n)>0$ forces both $d/2$ and $d/2+1$ to lie within these intervals, thus d can take any negative value except $\{...,-6,-4,-2\}$, the set of even negative integers. When $d<0$, both $V(R_n) < V(R_{n-1})$ and $A(R_n) < A(R_{n-1})$.

If $n \geq 1$, the probability to move outside the annulus number n is

$$p_n = A(R_n) / (A(R_n) + A(R_{n-1})),$$

while the probability to move inside this annulus is $q_n = 1 - p_n$. If $n = 0$, we assume that the probability to leave the central ball of radius x_0 is $p_0 = 1$. See [2], [3].

Note that, if $d > 1$, $p_n > 1/2$ if $n \geq 1$, while if $d < 1$, $p_n < 1/2$ if $n \geq 1$.

Equivalently, $p_0 = 1$ and for $n \geq 1$

$$\begin{aligned} p_n &= \frac{(n+x_0)^{d-1}}{(n+x_0)^{d-1} + (n-1+x_0)^{d-1}} \\ q_n &= \frac{(n+x_0-1)^{d-1}}{(n+x_0)^{d-1} + (n-1+x_0)^{d-1}} = 1 - p_n. \end{aligned}$$

are the transition probabilities of this random walk on $\mathbb{Z}_+ = \{0, 1, ...\}$. It is reflected at the origin.

Suppose we deal with a random walk with $d > 1$ (with $A(R_n)$ expanding). Consider the transformation $d \to d' = 2 - d < 1$ with $A'(R_n) = \frac{2\pi^{d'/2}}{\Gamma(d'/2)} R_n^{d'-1}$ contracting. Then $(n \geq 1)$

$$\begin{aligned} p_n &\to p'_n = q_n = \frac{(n+x_0-1)^{d-1}}{(n+x_0)^{d-1} + (n-1+x_0)^{d-1}} \\ q_n &\to q'_n = p_n = \frac{(n+x_0)^{d-1}}{(n+x_0)^{d-1} + (n-1+x_0)^{d-1}} = 1 - p'_n. \end{aligned}$$

The Markov chain with transition probabilities $p'_0 = 1$ and $(p'_n, q'_n)_{n\geq 1}$ is thus the Wall dual to the Markov chain with transition probabilities $p_0 = 1$ and $(p_n, q_n)_{n \geq 1}$, see [8]. And the random walk model makes sense for all d.

The probability sequence p_n, $n \geq 1$ is monotone decreasing if $d > 1$, while it is monotone increasing if $d < 1$. We have

$$p_n \sim \frac{1}{2}\left(1 + \frac{d-1}{2(n+x_0)}\right) \text{ as } n \to \infty,$$

so $p_n \to 1/2$ as $n \to \infty$ either from above $(d > 1)$ or from below $(d < 1)$ and the corrective term is $O(1/n)$.

We suppose $\overline{p}_0 = 1$ and we look for an homographic model for the transition probabilities

$$\begin{aligned} \overline{p}_n &= \frac{n+x_0+a}{2(n+x_0+b)} = \frac{n+x_0+a}{(n+x_0+a) + (n+x_0+a+2(b-a))} \\ \overline{q}_n &= 1 - \overline{p}_n,\ n \geq 1, \end{aligned}$$

which are the closer possible to the original ones. Of course the parameters (a, b) will then depend on (x_0, d).

To do this, we impose $\overline{p}_n \sim p_n$ as $n \to \infty$ and $\overline{p}_1 = p_1$. This leads to

$$a = \frac{(3 + 2x_0 - d)p_1 - (1 + x_0)}{1 - 2p_1} \text{ and } b = a - \frac{d-1}{2}.$$

Under these hypothesis, the models p_n and $\overline{p}_n$ agree fairly well (ranging from 10^{-5} to 10^{-2}), for all $n \geq 0$ and all $x_0 > 0$ and d. When $d = 1$ or $d = 2$, the two models are even exactly the same ($p_n = \overline{p}_n = 1/2$, $n \geq 1$ in the first case, $p_n = \overline{p}_n = (n + x_0) / (2n + 2x_0 - 1)$, $n \geq 1$ in the second case, obtained while $a = 0$ and $b = -\frac{1}{2}$).

If $x_0 \to 0$, the model makes sense only if $d \geq 1$ and then $p_1 \to 1$ and so $a \to d-2$; as a result, $\overline{p}_n = (n + d - 2) / (2n + d - 3)$, $n \geq 1$. Note $\overline{p}_1 = 1$, see [2].

Suppose $d > 1$. The homographic model $\overline{p}_n$ may be written as

$$\begin{aligned} \overline{p}_n &= \frac{n + \overline{x}_0}{(n + \overline{x}_0) + (n + \overline{x}_0 - (d-1))} \\ \overline{q}_n &= 1 - \overline{p}_n,\ n \geq 1, \end{aligned}$$

where $\overline{x}_0 = x_0 + a$, $a = a(x_0, d)$. Thus, with $\overline{R}_n = n + \overline{x}_0$ and $\overline{R}_{n-1} = n + \overline{x}_0 - (d-1)$, $\overline{p}_n = A\left(\overline{R}_n\right) / \left(A\left(\overline{R}_n\right) + A\left(\overline{R}_{n-1}\right)\right)$ with $A\left(\overline{R}_n\right) = 2\pi \overline{R}_n$, the circumference of a disk in dimension 2. Equivalently, $\overline{R}_n = \overline{x}_0 + (d-1)\, n$.

Under the transformation $d \to d' = 2 - d$, we have

$$\overline{p}_n \to \overline{p}'_n = \frac{n + \overline{x}'_0}{(n + \overline{x}'_0) + (n + \overline{x}'_0 - (d'-1))}$$

where $\overline{x}'_0 = x'_0 + a'$ with $x'_0 = x_0 + d - 1$ and

$$a' = \frac{(3 + 2x'_0 - d)\, p_1 - (1 + x'_0)}{1 - 2p_1};\ b' = a' + \frac{d-1}{2}.$$

7.2. **Special cases.** • $a = x_0$ and $d > 2$.

If we impose $a = x_0$ we get

$$x_0 (1 - 2p_1) = (3 + 2x_0 - d)\, p_1 - (1 + x_0)\,.$$

This is also

$$\left(\frac{x_0}{1 + x_0}\right)^{d-1} = 1 - \frac{d-1}{2x_0 + 1}.$$

There is a $x_0 =: x_0(d) \in (0, 1)$ obeying this equation only if $d > 2$ and then

$$\overline{p}_n = \frac{n + 2x_0}{2(n + 2x_0) - (d-1)} = \frac{1}{2}\left(1 + \frac{d-1}{2n + 4x_0 - (d-1)}\right),\ n \geq 1.$$

• $a = -x_0$ and $d < 2$. See [6].

If we impose $a = -x_0$ we get

$$-x_0 (1 - 2p_1) = (3 + 2x_0 - d)\, p_1 - (1 + x_0)\,.$$

This is also $p_1 = 1/(3-d)$. Thus

$$x_0 = 1/\left((2-d)^{-1/(d-1)} - 1\right)$$

which makes sense only if $d < 2$. In this case, $x_0 \in (0, 1)$ if $0 < d < 2$ and

$$\overline{p}_n = \frac{n}{2(n + b - a)} = \frac{1}{2}\left(1 + \frac{d-1}{2n - (d-1)}\right),\ n \geq 1,$$

which is independent of x_0. We note that this model is still valid, would dimension d be negative.

• $a = -x_0 + d - 1$ and $d > 1$. See [12].

If we impose $a = -x_0 + d - 1$, we get

$$(-x_0 + d - 1)(1 - 2p_1) = (3 + 2x_0 - d)\, p_1 - (1 + x_0).$$

This is also

$$\left(\frac{x_0}{1 + x_0}\right)^{d-1} = \frac{d - x_0}{1 + x_0}.$$

There is a $x_0 =: x_0(d) > 0$ obeying this equation only if $d > 1$ with $x_0 \in (0, 1)$ if $d < 2$, $x_0 \geq 1$ if $d > 2$ and then

$$\overline{p}_n = \frac{n + d - 1}{2n + (d - 1)} = \frac{1}{2}\left(1 + \frac{d - 1}{2n + (d - 1)}\right), \, n \geq 1,$$

which is independent of x_0. The latter two models are Wall duals.

7.3. **Thermodynamics.** In both cases of the Bessel random walk and the homographic random walk, we have $\lambda = (d-1)/2$ leading to $\mathrm{w} = (d-1)(3-d)/8$. The random walk is positive recurrent if $d < 0$ or $d > 4$ (corresponding to $\mathrm{w} < -3/8$) and null recurrent if $0 < d < 4$ (corresponding to $-3/8 < \mathrm{w} < 1/8$), [14].

In such random walk models, one can compute explicitly the b_n solving the recurrence $b_n b_{n+1} = \frac{1}{4p_n q_{n+1}}$, $n \geq 0$, together with the unique critical value of b_0 leading to $b_n \to 1$. Clearly the Pochhammer symbols are involved and making use of Stirling formula. We skip the details.

8. APPENDIX: PROOFS

We now come to the proofs of our statements.

$A.1$ PROOF of THEOREM 2.

(i) Assume $\alpha := \liminf_{n\to\infty} \frac{1}{\sqrt{b_n b_{n+1}}} > 0$. Let $\alpha > \epsilon > 0$ and N an integer such that for any $n \geq N$, $\frac{1}{\sqrt{b_n b_{n+1}}} \geq \alpha - \epsilon$. Let R_N be the matrix R without its N first rows and N first columns. For $i, j > N$ we have for any integer k, $(R^k)_{i,j} \geq (R_N^k)_{i,j} \geq (T^k)_{i-N,j-N}$ where T is the tridiagonal matrix with zeros on the diagonal and the other nonzero entries equal to $(\alpha - \epsilon)/2$. Since the number of walks of length $2k$ from $i - N$ to $j - N$ is 2^{2k} up to a polynomial correction in k, one gets

$$\lim_{k\to\infty} \left((T^{2k})_{i-N,j-N}\right)^{1/(2k)} = \alpha - \epsilon$$

and the lower bound follows. The proof of the upper bound is left to the reader.

$(i) \Rightarrow (ii)$

(iii) follows immediately from the fact that v_0 determines v_1, and for $n \geq 2$ we have a second order recursion equation for v_n as a function of v_{n-1} and v_{n-2}.

(iv) Assume there is a positive w solving (3.2). Let (v_n) be a solution of $Rv = \rho v$ with $v_1 > 0$. It follows that $w_1 v_2 - w_2 v_1 = 2v_1\sqrt{b_1 b_2}$. It follows from Lemma 14 that $\forall n \geq 2$, $\frac{v_{n+1}}{w_{n+1}} > \frac{v_n}{w_n}$. Since $v_2 = 2\rho v_1 \sqrt{b_1 b_2} > 0$, the result follows by recursion.□

• $A.2$ PROOF of LEMMA 3.

We start with the following proposition.

Proposition 13. *Let $1 \le \rho'$ be such that the equation $Rw' = \rho' w' - \mathbf{1}_{p=1}$ has a positive solution. Then for any $\rho > \rho'$, the equation $Rw = \rho w - \mathbf{1}_{p=1}$ has a positive solution.*

Proof: Let (w'_n) be a positive solution of

$$Rw' = \rho'\, w' - \mathbf{1}_{p=1}.$$

so that

$$\sigma_1(\rho') := 2\sqrt{b_1 b_2}\left(\rho' - \frac{1}{w'_1}\right) > 0$$

and for $n \ge 2$, let

$$\sigma_n(\rho') = \frac{w'_{n+1}}{w'_n}.$$

Note that this formula also holds for $n = 1$ with our definition of $\sigma_1(\rho')$.

Let $1 \le \rho' < \rho$. Consider the sequence $(\sigma_n) = (\sigma_n(\rho))$ defined by $\sigma_1(\rho) = \sigma_1(\rho')$, and recursively for $n \ge 2$ by

$$\sigma_n = 2\rho\sqrt{b_n b_{n+1}} - \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{1}{\sigma_{n-1}}.$$

We have

$$\frac{\partial \sigma_n}{\partial \rho} = 2\sqrt{b_n b_{n+1}} + \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{1}{\sigma_{n-1}^2}\frac{\partial \sigma_{n-1}}{\partial \rho}.$$

Hence, since $\partial_\rho \sigma_1(\rho) = \partial_\rho \sigma_1(\rho') = 0$ we conclude recursively that for all $n \ge 2$

$$\frac{\partial \sigma_n(\rho)}{\partial \rho} > 0 \text{ and } \sigma_n(\rho) > \sigma_n(\rho').$$

Hence the sequence w_n defined by

$$w_1 = \frac{1}{\rho - \rho' + \frac{1}{w'_1}} > 0, \tag{8.1}$$

$$w_2 = 2\sqrt{b_1 b_2}\,(\rho w_1 - 1) = \sigma_1(\rho') w_1 > 0,$$

and for $n \ge 3$

$$w_n = w_2 \prod_{j=2}^{n-1} \sigma_j(\rho)$$

is positive and satisfies

$$Rw = \rho w - \mathbf{1}_{p=1}$$

completing the proof of the Proposition. □

Therefore, letting w'_1 decrease to $\mathfrak{w}_1(\rho')$, we get $\mathfrak{w}_1(\rho) < \mathfrak{w}_1(\rho')$, since from (8.1)

$$\mathfrak{w}_1(\rho) \le \frac{1}{\rho - \rho' + \frac{1}{\mathfrak{w}_1(\rho')}} < \mathfrak{w}_1(\rho').$$

This fact proves (i) of Lemma 3 except continuity. (ii) and (iii) follow immediately. The proof of (iv), (v) and continuity in (i) rely on several results of independent interest. □

The following lemma is essentially due to Josef Hoëné-Wronski.

Lemma 14. *If v and w satisfy $(Rv)_n = \rho v_n$, $(Rw)_n = \rho w_n$, for $n \geq k \geq 2$, then for $n \geq k$*

$$v_{n+1}w_n - w_{n+1}v_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}}\,(v_n w_{n-1} - w_n v_{n-1})\,.$$

Hence

$$v_{n+1}w_n - w_{n+1}v_n = \left(\prod_{j=k}^{n} \frac{b_{j+1}}{b_{j-1}}\right)^{1/2} (v_k w_{k-1} - w_k v_{k-1})\,.$$

Proof: From $(Rv)_n = \rho v_n$, it holds that

$$\frac{v_{n+1}}{v_n} + \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{v_{n-1}}{v_n} = 2\rho\sqrt{b_n b_{n+1}}$$

and similarly for w_n. The difference between the two identities gives the result. □

For $\rho > 1$, we will denote by $x_+ \geq x_-$ ($x_+(\rho) \geq x_-(\rho)$) the two (real) solutions of

$$x^2 - 2\rho x + 1 = 0. \tag{8.2}$$

Note that $0 < x_- < 1 < x_+$.

Proposition 15. *For $\rho > 1$, the equation*

$$(Rw)_n = \rho w_n$$

for all $n \geq 2$ has two independent solutions $w^{\pm}$ such that

$$w_n^{\pm} \sim x_{\pm}^n.$$

Any other solution is a linear combination of these two solutions.

Note that these solutions may not be positive.

Remark. The heuristics is clear: one tries an ansatz $w_n = x^n$ and one chooses the value of x such that the equation $(Rw)_n = \rho w_n$ is satisfied for large n at dominant order.

Proof: The equation $(Rw)_n = \rho w_n$ for $n \geq 2$ is a linear recursion of order two, therefore the set of solutions is a vector space of dimension two. We first construct a solution w^- using an idea of Levinson [15].

For $n > 1$ we have

$$\frac{w_{n+1}}{2\sqrt{b_n\, b_{n+1}}} + \frac{w_{n-1}}{2\,\sqrt{b_n\, b_{n-1}}} = \rho w_n$$

which can be rewritten (with $p = n - 1$)

$$w_p = 2\rho\sqrt{b_{p+1}b_p}\,w_{p+1} - \sqrt{\frac{b_p}{b_{p+2}}}\,w_{p+2}.$$

Let $u_p = w_p/x_-^p$, we get

$$u_p = 2\rho x_-\sqrt{b_{p+1}b_p}\,u_{p+1} - x_-^2\sqrt{\frac{b_p}{b_{p+2}}}\,u_{p+2}.$$

Let $\delta_p = u_p - 1$, we get

$$\delta_p = r_p + 2\rho x_- \sqrt{b_{p+1} b_p}\delta_{p+1} - x_-^2 \sqrt{\frac{b_p}{b_{p+2}}}\delta_{p+2}.$$

with

$$r_p = 2\rho x_- \sqrt{b_{p+1} b_p} - x_-^2 \sqrt{\frac{b_p}{b_{p+2}}} - 1.$$

This can be rewritten

$$\delta_p - 2\rho x_- \delta_{p+1} + x_-^2 \delta_{p+2} = r_p + T\left(\delta\right)_p \tag{8.3}$$

where T is the operator defined by

$$T\left(\delta\right)_p = 2\rho x_- \left(\sqrt{b_{p+1} b_p} - 1\right)\delta_{p+1} - x_-^2 \left(\sqrt{\frac{b_p}{b_{p+2}}} - 1\right)\delta_{p+2}.$$

We now consider the operator defined by

$$U\left(s\right)_p = x_-^{-2p} \sum_{j=p}^{\infty} s_j \frac{1 - x_-^{2(j+1)}}{1 - x_-^2}.$$

Using hypothesis (3.3) it is easy to verify that there exists $N > 4$ large enough such that the linear operator $U \circ T$ is bounded with norm less than $1/2$ in the Banach space $c^0([N, \infty))$,

$$c^0([N, \infty)) = \left\{(u)_{n \geq N} : \lim_{n\to\infty} |u_n| = 0\right\}.$$

It is well known that equipped with the sup norm, $c^0([N, \infty))$ is a Banach space (see for example [22]). Similarly, using

$$r_p = 2\rho x_- \left(\sqrt{b_{p+1} b_p} - 1\right) - x_-^2 \left(\sqrt{\frac{b_p}{b_{p+2}}} - 1\right),$$

and hypothesis (3.3) we deduce $U\left(r\right) \in c^0([N, \infty))$. Taking N larger if necessary, we can also assume that

$$\|U\left(r\right)\|_{c^0([N,\infty))} < 1/4.$$

Therefore the sequence $(\tilde{\delta})_{[N,\infty)}$ defined by

$$\tilde{\delta} = (I - U \circ T)^{-1} U\left(r\right)$$

has norm at most $1/2$ in $c^0([N, \infty))$. It is easy to verify that for any $p \geq N$, this sequence satisfies equation (8.3). For $p \geq N$ we define

$$w_p^- = x_-^p (1 + \tilde{\delta}_p).$$

For $1 \leq p < N$, w_p^- is defined recursively (downward) using again (8.3). We obviously have, since $\tilde{\delta} \in c^0([N, \infty))$,

$$\lim_{n\to\infty} \frac{w_n^-}{x_-^n} = 1$$

and $(Rw^-)_p = \rho w_p^-$ for $p \geq 2$.

For $n \geq N$ define (the idea comes from the Wronskian, see Lemma 14)

$$w_n^+ = C w_n^- \sum_{j=4}^{n-1} \frac{1}{w_j^- \, w_{j-1}^-} \left(\prod_{l=2}^{j} \frac{b_{l+1}}{b_{l-1}} \right)^{1/2},$$

where C is a positive constant. For $n < N$, we define w_n^+ recursively downward. It is easy to verify (using $0 < x_- = 1/x_+ < 1$) that one can choose the positive constant C such that

$$\lim_{n\to\infty} \frac{w_n^+}{x_+^n} = 1.$$

Moreover, we have $(Rw^+)_p = \rho w_p^+$ for $p \geq 2$. □

Lemma 16. *If $\rho > 1$ and the equation*

$$Rw = \rho w - \mathbf{1}_{p=1}$$

has a positive solution, then for p large, the positive solution defined in (3.4) obeys

$$\mathfrak{w}_p \sim x_-^p$$

and for $\mathfrak{v}$ the positive solution of $R\mathfrak{v} = \rho\mathfrak{v}$ with $\mathfrak{v}_1 = 1$ we have

$$\mathfrak{v}_n \sim x_+^n.$$

Proof: From Proposition 15 we have for some constants A and B and for $n \geq 2$

$$\mathfrak{w}_n = A w_n^+ + B w_n^-.$$

Assume $A \neq 0$ (otherwise the result follows from Proposition 15). From the positivity of $\mathfrak{w}$ we have if $A \neq 0$

$$\mathfrak{w}_n > c x_+^n$$

for some $c > 0$ and any $n \geq 1$. From the same Proposition 15 we conclude that there exists a number $\Gamma > 0$ such that for any $n \geq 1$

$$0 \leq \mathfrak{v}_n \leq \Gamma x_+^n.$$

Therefore, the sequence (w) defined for $n \geq 1$ by

$$w_n = \mathfrak{w}_n - \frac{c}{2\Gamma} \mathfrak{v}_n$$

is a positive solution of

$$Rw = \rho w - \mathbf{1}_{n=1}$$

which satisfies

$$w_1 = \mathfrak{w}_1 - \frac{c}{2\Gamma} < \mathfrak{w}_1$$

which is a contradiction. Therefore $A = 0$ and this proves the first part of the statement. For the second part, applying again Proposition 15, we have to exclude that $\mathfrak{v}_n \sim x_-^n$. Assume this is the case. Using Lemma 14 for $\mathfrak{w}$ and $\mathfrak{v}$, and the asymptotic of $\mathfrak{w}_n$ we would get

$$\left(\prod_{j=1}^{n} \frac{b_{j+1}}{b_{j-1}} \right)^{1/2} \sim x_-^{2n}$$

which is a contradiction since $x_- < 1$ and (b_n) converges to one. □

Lemma 17. *Assume that for some $\rho > 1$, the equation*

$$Rv = \rho v$$

has a positive solution $\mathfrak{v}$ which satisfies $\mathfrak{v}_1 = 1$, and

$$\mathfrak{v}_n \sim x_+(\rho)^n.$$

Then there exists $\epsilon > 0$ such that for any $\rho' \in [\rho - \epsilon, \rho + \epsilon]$, the equation

$$Rv = \rho' v$$

has a positive solution such that

$$\mathfrak{v}_n(\rho') \sim x_+(\rho')^n.$$

Proof: The sequence $\sigma_n(\rho) = \mathfrak{v}_{n+1}(\rho)/\mathfrak{v}_n(\rho)$ (defined for $n \geq 1$) satisfies for $n \geq 2$

$$\sigma_n(\rho) = 2\rho\sqrt{b_n b_{n+1}} - \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{1}{\sigma_{n-1}(\rho)}.$$

Moreover, by Lemma 16, this sequence converges to $x_+(\rho)$ when n tends to infinity.

For $\rho > 1$ since $x_+(\rho) > 1$ we can choose $\delta > 0$ such that $\delta < x_+(\rho)/2$, and

$$0 < \delta < x_+(\rho) - \frac{1}{x_+(\rho)}.$$

Note that

$$0 < \frac{\delta}{x_+(\rho)\left(x_+(\rho) - \delta\right)} < \delta.$$

Choose $0 < \delta' < \delta$ such that

$$0 < \delta' < \delta - \frac{\delta}{x_+(\rho)\left(x_+(\rho) - \delta\right)}.$$

Since (b_n) converges to 1, and $\sigma_n(\rho)$ converges $x_+(\rho) > 1$, one can find N large enough such that

$$\inf_{n>N} \sigma_{n-1}(\rho) > \delta$$

and

$$\sup_{n>N}\left(2\delta'\sqrt{b_n b_{n+1}} + \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{\delta}{\sigma_{n-1}(\rho)\left(\sigma_{n-1}(\rho) - \delta\right)}\right) \leq \delta. \tag{8.4}$$

By continuity, for any ρ' with $|\rho' - \rho|$ small enough and any σ'_1 with $|\sigma'_1 - \sigma_1|$ small enough we can define recursively a sequence $(\sigma'_{1\leq n\leq N})$ such that

$$\inf_{1\leq n\leq N} \sigma'_n > 0,\ |\sigma'_N - \sigma_N| < \delta,$$

and for any $2 \leq n \leq N$

$$\sigma'_n = 2\rho'\sqrt{b_n\, b_{n+1}} - \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{1}{\sigma'_{n-1}}. \tag{8.5}$$

We now observe that if $|\rho' - \rho| < \delta'$ and if for some $n \geq N+1$, σ'_{n-1} is defined and satisfies $|\sigma'_{n-1} - \sigma_{n-1}| < \delta$, then σ'_n defined by (8.5) satisfies also $|\sigma'_n - \sigma_n| < \delta$ since

$$|\sigma'_n - \sigma_n| = \left|2(\rho' - \rho)\sqrt{b_n b_{n+1}} + \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{\sigma'_{n-1} - \sigma_{n-1}}{\sigma'_{n-1}\,\sigma_{n-1}}\right|$$

$$\leq 2\delta'\sqrt{b_n b_{n+1}} + \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{\delta}{(\sigma_{n-1}-\delta)\sigma_{n-1}} < \delta,$$

by condition (8.4). This implies in particular $\sigma'_n > 0$. Therefore we can define recursively for all $n \geq 1$ a sequence $(\sigma'_n) > 0$ satisfying (8.5). We leave to the reader to prove that (σ'_n) converges exponentially fast to $x_+(\rho')$. It then follows that the sequence v'_n defined recursively by $v'_0 = 1$ and

$$v'_{n+1} = \sigma'_n v'_n$$

is a positive solution of

$$Rv' = \rho' v'$$

with

$$v'_n \sim x_+(\rho')^n.$$

The Lemma is proved by taking $\epsilon = \delta'$. □

Lemma 18. *Assume that for some $\rho > 1$, the equation*

$$Rv = \rho v$$

has a positive solution $\mathfrak{v}$ which satisfies $\mathfrak{v}_1 = 1$, and

$$\mathfrak{v}_n \sim x_+(\rho)^n.$$

Then the equation

$$Rw = \rho w - \mathbf{1}_{n=1}$$

has a positive solution. Moreover

$$\lim_{n\to\infty} \frac{\mathfrak{w}_{n+1}}{\mathfrak{w}_n} = x_-(\rho)\,.$$

Proof: Let w be a solution of the equation

$$Rw = \rho w - \mathbf{1}_{n=1},$$

and v a solution of

$$Rv = \rho v,$$

with $v_1 = 1$. We have

$$w_2 v_1 - v_2 w_1 = 2\sqrt{b_1 b_2}\,(\rho w_1 - 1) - 2\sqrt{b_1 b_2}\rho w_1 = -2\sqrt{b_1 b_2}.$$

Therefore from Lemma 14 (with $k = 2$) we get for any $n \geq 2$

$$w_n v_{n-1} - v_n w_{n-1} = -2\sqrt{b_1 b_2}\left(\prod_{j=3}^{n} \frac{b_{j+1}}{b_{j-1}}\right)^{1/2}.$$

This implies for any $n \geq 2$

$$\frac{w_n}{v_n} = \frac{w_{n-1}}{v_{n-1}} - 2\frac{\sqrt{b_1\, b_2}}{v_n\, v_{n-1}}\left(\prod_{j=3}^{n} \frac{b_{j+1}}{b_{j-1}}\right)^{1/2}.$$

Therefore for $n \geq 2$

$$\frac{w_n}{v_n} = w_1 - 2\sum_{q=2}^{n} \frac{\sqrt{b_1\, b_2}}{v_q v_{q-1}}\left(\prod_{j=3}^{q} \frac{b_{j+1}}{b_{j-1}}\right)^{1/2}.$$

We now take $v = \mathfrak{v}$. We conclude that for

$$w_1 \ge 2 \sum_{q=2}^{\infty} \frac{\sqrt{b_1\, b_2}}{\mathfrak{v}_q \mathfrak{v}_{q-1}} \left(\prod_{j=3}^{q} \frac{b_{j+1}}{b_{j-1}} \right)^{1/2}$$

we have a positive solution (w).

It is easy to verify that for

$$w_1 = 2 \sum_{q=2}^{\infty} \frac{\sqrt{b_1\, b_2}}{\mathfrak{v}_q \mathfrak{v}_{q-1}} \left(\prod_{j=3}^{q} \frac{b_{j+1}}{b_{j-1}} \right)^{1/2} ,$$

we have

$$\lim_{p\to\infty} \frac{w_{p+1}}{w_p} = x_- .$$

It follows from Proposition 15 that $\mathfrak{w} = w$.

We now give the proof of (iv) in Lemma 3. Assume $\rho_*(R) > 1$ and

$$\lim_{\rho \searrow \rho_*(R)} \mathfrak{w}_1(\rho) < \infty .$$

It follows from Lemma 16 that $\mathfrak{v}_n \sim x_+^n(\rho)$. We can now apply Lemma 17 and then Lemma 18 to conclude that for some $1 < \rho' < \rho_*(R)$, the equation $Rw = \rho' w - \mathbf{1}_{n=1}$ has a positive solution which contradicts the definition of $\rho_*(R)$. $\square$

Proposition 19. *For any $n \ge 1$, the function $\mathfrak{w}_n(\rho)$ is monotone decreasing in $\rho \in (\rho_*, \infty)$ if either $\rho_* > 1$ or $\rho_* = 1$ and R is $1-$recurrent. $\mathfrak{w}_n(\rho)$ is monotone decreasing in $\rho \in [1, \infty)$ if $\rho_* = 1$ and R is $1-$transient.*

Proof: Let $\rho' > \rho > \rho_*$ if either $\rho_* > 1$ or $\rho_* = 1$ and R is $1-$transient or $\rho' > \rho \ge 1$ if $\rho_* = 1$ and R is $1-$transient. Let $w = \mathfrak{w}(\rho)$ and $w' = \mathfrak{w}(\rho')$.

From $Rw = \rho w - \mathbf{1}_{p=1}$ and $Rw' = \rho' w' - \mathbf{1}_{p=1}$, we conclude that

$$R(w - w') = \rho(w - w') - (\rho' - \rho) w' .$$

The sequence $s_n = (w_n - w'_n)/w_n$ satisfies

$$\frac{1}{w_n} (R(ws))_n - \rho s_n = -(\rho' - \rho) w'_n / w_n$$

where $(ws)_n = w_n s_n$. Assume the sequence $s_n = (w_n - w'_n)/w_n$ takes negative values. We now derive a contradiction using the so called negative minimum principle. Using Lemma 3.(i) we have $w_1 - w'_1 \ge 0$. From Lemma 16 if $\rho > 1$ and Corollary 23 below if $\rho = 1$, we conclude that for any n large enough $w_n - w'_n > 0$. Therefore there can be only a finite number of indices n such that $s_n < 0$. Let n_* be an index (there may be several) such that

$$s_{n_*} = \inf_n s_n < 0 .$$

Note that $n_* > 1$. Since $s_{n_*+1} \ge s_{n_*}$ and $s_{n_*-1} \ge s_{n_*}$ we conclude that

$$0 > -(\rho' - \rho) w'_{n_*} / w_{n_*} = \frac{w_{n_*+1} s_{n_*+1}}{2 w_{n_*} \sqrt{b_{n_*}\, b_{n_{*+1}}}} + \frac{w_{n_*-1} s_{n_*-1}}{2 w_{n_*} \sqrt{b_{n_*} b_{n_*-1}}} - \rho s_{n_*}$$

$$\ge s_{n_*} \left(\frac{w_{n_*+1}}{2 w_{n_*} \sqrt{b_{n_*} b_{n_{*+1}}}} + \frac{w_{n_*-1}}{2\, w_{n_*} \sqrt{b_{n_*} b_{n_*-1}}} - \rho \right) = 0,$$

a contradiction which proves the announced monotonicity. $\square$

We now give the proof of continuity stated in (i) of Lemma 3 for $\rho > \rho_*$. The proof is by contradiction. Assume there exists $\tilde{\rho} > \rho_*$ such that

$$\lim_{\rho \nearrow \tilde{\rho}} \mathfrak{w}_1(\rho) > \lim_{\rho \searrow \tilde{\rho}} \mathfrak{w}_1(\rho).$$

Here we used the monotonicity of $\mathfrak{w}_1(\rho)$ proved earlier. By continuity, it follows that the two sequences defined for $n \geq 1$ by

$$\overline{\mathfrak{w}}_n = \lim_{\rho \nearrow \tilde{\rho}} \mathfrak{w}_n(\rho)$$

and

$$\underline{\mathfrak{w}}_n = \lim_{\rho \searrow \tilde{\rho}} \mathfrak{w}_n(\rho)$$

satisfy the equation

$$Rw = \tilde{\rho} w - \mathbf{1}_{n=1}.$$

From Proposition 19, we conclude that for any $\tilde{\rho} > \rho' > \rho_*$ we have for all $n \geq 1$

$$\mathfrak{w}_n(\rho') \geq \overline{\mathfrak{w}}_n \geq \underline{\mathfrak{w}}_n \geq 0.$$

Therefore

$$\lim_{n \to \infty} \overline{\mathfrak{w}}_n = \lim_{n \to \infty} \underline{\mathfrak{w}}_n = 0.$$

We get a contradiction from Proposition 15 and Lemma 16 since the two sequences $(\underline{\mathfrak{w}}_n)$ and $(\overline{\mathfrak{w}}_n)$ must be linearly independent.

Finally we give the proof of (v) in Lemma 3. Monotonicity has already been proved in Proposition 19. Let

$$\tilde{\mathfrak{w}}_1 = \lim_{\rho \searrow 1} \mathfrak{w}_1(\rho).$$

By continuity, $\tilde{\mathfrak{w}} = \lim_{\rho \searrow 1} \mathfrak{w}$ is a solution of $R\,\tilde{\mathfrak{w}} = \tilde{\mathfrak{w}} - \mathbf{1}_{n=0}$. Therefore $\mathfrak{w}_1(1) < \infty$. By Proposition 19, we have for any n, $\tilde{\mathfrak{w}}_n \leq \mathfrak{w}_n(1)$, hence equality follows by the minimality in the definition of $\mathfrak{w}$. □

• $A.3$ PROOF of PROPOSITION 4.

Proof: It is easy to verify that if $b_n = 1$ for all $n \geq 1$, then $w_n = 2$ satisfies $Rw = w - \mathbf{1}_{n=1}$, so that $\rho_* = 1$. Recall that in [19], R is called $1-$transient if

$$\sum_n (R^n)_{1,1} < \infty.$$

The left hand side is obviously a decreasing function in each b_n, and $\rho_*(R) \leq 1$. The proof follows using (ii) of Theorem 2 and (i) of Theorem 1. □

• $A.4$ PROOF of PROPOSITION 5.

Before we start the proof we need some preliminary results. The first goal is to obtain an analog of Lemma 16 when $\rho_*(R) = 1$. We define a family of Banach spaces $\mathfrak{B}_{n_0,\gamma}$ which depend on an integer $n_0 > 1$ and a positive number γ by

$$\mathfrak{B}_{n_0,\gamma} = \left\{ (u)_{n \geq n_0} \, : \, \sup_{n \geq n_0} |u_n| \cdot n^{\gamma} < \infty \right\}.$$

It is easy to verify that $\mathfrak{B}_{n_0,\gamma}$ is a Banach space when equipped with norm

$$\|(u_n)\|_{n_0,\gamma} = \sup_{n \geq n_0} |u_n| \cdot n^{\gamma}.$$

Lemma 20. *For $\alpha \in \mathbb{C}$ ($\Re(\alpha) \le 1/2$), there exists $N_0 = N_0(\alpha)$ such that the operator $\mathcal{S}$ defined by*

$$(\mathcal{S}h)_n = \sum_{m=n}^{\infty} \sum_{p=m+1}^{\infty} \frac{h_p}{1-\frac{\alpha}{p}} \cdot \prod_{j=m+1}^{p-1} \frac{j+\alpha}{j-\alpha}$$

is a bounded operator from $\mathfrak{B}_{N,\gamma}$ to $\mathfrak{B}_{N-1,\gamma-2}$ for any $\gamma > 2$ and $N > N_0$ with a norm uniformly bounded in N.

The proof is left to the reader.

Lemma 21. *For any $\alpha \in \mathbb{C}$ ($\Re(\alpha) \le 1/2$), let $\mathcal{T}$ be the operator defined by*

$$(\mathcal{T}u)_n = u_{n+1} + u_{n-1} - 2u_n + \frac{\alpha}{n}(u_{n+1} - u_{n-1}).$$

There exists $N_0 = N_0(\alpha) > 0$ such that for any $N > N_0(\alpha)$ and $\gamma > 2$, $\mathcal{T}$ is a bounded operator from $\mathfrak{B}_{N-1,\gamma}$ to $\mathfrak{B}_{N,\gamma}$ with norm uniformly bounded in N. Moreover $\mathcal{S} \circ \mathcal{T}$ is the canonical injection from $\mathfrak{B}_{N-1,\gamma}$ to $\mathfrak{B}_{N-1,\gamma-2}$.

Proof: The first part of the statement is obvious. Let $u \in \mathfrak{B}_{N-1,\gamma}$, and let $g = \mathcal{T}u \in \mathfrak{B}_{N,\gamma}$. Letting

$$z_n = u_n - u_{n-1}, \tag{8.6}$$

we have

$$z_{n+1} - z_n + \frac{\alpha}{n}(z_{n+1} + z_n) = g_n.$$

In other words

$$z_{n+1}\left(1 + \frac{\alpha}{n}\right) = z_n\left(1 - \frac{\alpha}{n}\right) + g_n,$$

which can be rewritten

$$z_n = \frac{1+\frac{\alpha}{n}}{1-\frac{\alpha}{n}} z_{n+1} - \frac{g_n}{1-\frac{\alpha}{n}}.$$

We will use this relation only for $n > \alpha + 1$. We use the solution

$$z_n = -\sum_{p=n}^{\infty} \frac{g_p}{1-\frac{\alpha}{p}} \prod_{j=n}^{p-1} \frac{j+\alpha}{j-\alpha},$$

where for $p = n$ the product is equal to one. Note that this is well defined from the assumption on the decay of g_p and $\Re\alpha \le 1/2$.

From this sequence we can recover u_n by solving (8.6). We get

$$u_n = \sum_{m=n}^{\infty} \sum_{p=m+1}^{\infty} \frac{g_p}{1-\frac{\alpha}{p}} \prod_{j=m+1}^{p-1} \frac{j+\alpha}{j-\alpha} = (\mathcal{S}g)_n.$$

The result follows from Lemma 20. □

For w $< 1/8$, let $\alpha_+ > \alpha_-$ be the two solutions of the equation

$$\alpha^2 - \alpha + 2\mathrm{w} = 0. \tag{8.7}$$

Note that $\alpha_- < 1/2 < \alpha_+$.

For w $> 1/8$, the two solutions of $\alpha^2 - \alpha + 2\,\mathrm{w} = 0$ have real part equal to $1/2$ and we denote by α_- the solution with negative imaginary part. For w $= 1/8$ we define $\alpha_- = 1/2$.

Proposition 22. *Assume*

$$b_n = 1 - \frac{\mathrm{w}}{n^2} + \mathcal{O}(1)n^{-2-\zeta},$$

for some $\zeta > 0$ *and* $\mathrm{w} \neq 1/8$. *Then, the equation*

$$(Rv)_n = v_n$$

has two independent solutions $v^{\pm}$ *such that for large* n

$$v_n^{\pm} = n^{\alpha_\pm}\left(1 + o(1)\right).$$

For $\mathrm{w} = 1/8$, *we have*

$$v_n^- = n^{1/2}\cdot\left(1+o(1)\right),\ v_n^+ = n^{1/2}\log n \cdot\left(1+o(1)\right).$$

Any solution of $(Rv)_n = v_n$ *for large* n *is a linear combination of* $v^{\pm}$.

Note that these solutions may not be positive.

Remark. The heuristics is clear: one tries an ansatz $v_n = n^{\alpha}$ and one chooses the value of α such that the equation $(Rv)_n = v_n$ is satisfied for large n at dominant order.

Proof: For $\alpha = \alpha_-$, we look for a solution of $Rv = v$ of the form

$$v_n = n^{\alpha}(1 + u_n)$$

with u_n small for large n. Using Proposition 34 below, it follows that there is a solution of $Rv = v$ satisfying $v_n^- = n^{\alpha_-}\left(1 + o\left(1\right)\right)$.

Let (w_n) be a solution of $Rw = w$ independent of v^-. From Lemma 14 we have

$$\frac{w_{n+1}}{v_{n+1}^-} - \frac{w_n}{v_n^-} = \frac{C_n}{v_n^- v_{n+1}^-}.$$

where C_n is a sequence converging to a non zero limit. Therefore, for $n > N + 1$

$$\frac{w_n}{v_n^-} = \frac{w_N}{v_N^-} + \sum_{p=N}^{n-1}\frac{C_p}{v_p^- v_{p+1}^-}.$$

Since

$$v_p^- = p^{\alpha_-}(1 + o(1))$$

with $\Re\alpha_- \leq 1/2$, the sum on the right hand side diverges or oscillates, and we get if $\alpha_- \neq 1/2$

$$0 < \lim_{n\to\infty}\left|\frac{w_n}{n^{\alpha_+}}\right| < \infty,$$

and the result follows for $\mathrm{w} \neq 1/8$ since

$$0 < \lim_{n\to\infty}\left|n^{\alpha_- - \alpha_+}\sum_{p=1}^{n}\frac{1}{p^{2\alpha_-}}\right| < \infty.$$

The same argument can be used for $\mathrm{w} = 1/8$ since $\lim_{n\to\infty}\frac{1}{\log n}\sum_{p=1}^{n}\frac{1}{p} = C$, Euler's constant.

Finally, since the equation $Rv = v$ can be solved by a recursion of order 2, its set of solution is a linear space of dimension two. □

Corollary 23. *Assume (3.5) holds with* $\mathrm{w} < 1/8$*. Assume that* $\rho = 1$ *and the equation*

$$Rw = \rho w - \mathbf{1}_{p=1}$$

has a positive solution. Then for p *large*

$$\mathfrak{w}_p \sim p^{\alpha_-}$$

and for $\mathfrak{v}$ *the positive solution of* $R\mathfrak{v} = \rho\mathfrak{v}$ *with* $\mathfrak{v}_1 = 1$ *we have*

$$\mathfrak{v}_n \sim n^{\alpha_+}.$$

Proof: The proof is similar to the proof of Lemma 16 and left to the reader.

We now start proving Proposition 5.

• **Proof of** (i) **in Proposition** 5.

If $\rho_*(R) > 1$ this follows from (iii) in Theorem 1. If $\rho_*(R) = 1$ and $\mathrm{w} > 1/8$, assume there exists a positive solution to

$$Rw = w - \mathbf{1}_{p=1}.$$

For $n \geq 2$ we have $(Rw)_n = w_n$ and Proposition 22 implies that there exist two constants A and B such that for $\sigma = \Im\alpha_+$

$$w_n = Av_n^+ + Bv_n^- = n^{1/2}\left(An^{i\sigma} + Bn^{-i\sigma}\right) + o\left(n^{1/2}\right).$$

It is easy to verify that there is no choice of (complex) A and B such that the right hand side is positive for any n. □

• **Proof of** (ii) **in Proposition** 5.

For $\mathrm{w} < 1/8$, let

$$b_n = 1 - \frac{\mathrm{w}}{n^2},\ \forall n \geq 3.$$

Choose $b_1 > 0$ and $b_2 > 0$ such that $4b_1b_2 < 1$.

Assume R is $1-$transient, namely there exists a $(w_n)_{n\geq 1}$ positive such that

$$Rw = w - \mathbf{1}_{n=1}.$$

We have

$$\frac{w_2}{2\sqrt{b_1b_2}} = w_1 - 1$$

$$\frac{w_3}{2\sqrt{b_3b_2}} + \frac{w_1}{2\sqrt{b_1b_2}} = w_2$$

hence

$$\frac{w_3}{2\sqrt{b_3b_2}} + \frac{1}{2\sqrt{b_1b_2}} = w_2\left(1 - \frac{1}{4b_1b_2}\right) < 0,$$

a contradiction. □

• **Proof of** (iii) **in Proposition** 5.

An example is given by the hypergeometric solution of Section 6.

• **Proof of** (iv) **in Proposition** 5.

For $\mathrm{w} < 0$, the matrix R is $1-$transient by Lemma 4.

For $0 \leq \mathrm{w} < 1/8$, we will use a continued fraction result. We will use Henrici's notation

$$\underset{n=1}{\overset{\infty}{\Phi}} \frac{f_j}{g_j}$$

for the continued fraction

$$\frac{f_1}{g_1 + \frac{f_2}{g_2 + \dots}}.$$

We have (for example from [10] formula **12.1-11**)

$$\underset{n=1}{\overset{\infty}{\Phi}} \frac{-\sqrt{b_{j+2}/b_j}}{-2\sqrt{b_{j+1}b_{j+2}}} \approx \underset{j=1}{\overset{\infty}{\Phi}} \frac{a_j}{1}$$

where

$$a_1 = \frac{1}{2\sqrt{b_1 b_2}}$$

and for $j > 1$

$$a_j = -\frac{1}{4 b_j b_{j+1}}.$$

For $\mathrm{w} > 0$ small enough we have for all $j \geq 2$

$$\left| a_j + \frac{1}{4} \right| \leq \frac{1}{4(4j^2 - 1)}.$$

By a result of Pringsheim (see for example [11]), this implies convergence of

$$\underset{j=2}{\overset{\infty}{\Phi}} \frac{a_j}{1}.$$

This implies that for $\mathrm{w} > 0$ small enough, the sequence $(\sigma_p)_{p \geq 1}$ defined for $p \geq 1$ by

$$\sigma_p = \underset{j=p}{\overset{\infty}{\Phi}} \frac{a_j}{1} = \underset{j=p}{\overset{\infty}{\Phi}} \frac{-\sqrt{b_{j+2}/b_j}}{-2\sqrt{b_{j+1}b_{j+2}}}$$

is well defined and continuous in w. Since it is nonnegative at $\mathrm{w} = 0$, by continuity it is nonnegative in a neighborhood of $\mathrm{w} = 0$.

By continuity in w we also have that for $\mathrm{w} > 0$ small enough,

$$\sigma_1 < 2\sqrt{b_1 b_2}.$$

We now define recursively a positive sequence $(w_n)_{n \geq 1}$ by

$$w_1 = \frac{1}{1 - \frac{\sigma_1}{2\sqrt{b_1 b_2}}}$$

and for $n > 1$

$$w_n = w_{n-1} \sigma_{n-1}.$$

It is easy to verify that the sequence $(\sigma_n)_{n \geq 1}$ obeys the same recursion as in Appendix $A.2$ and that the positive sequence $(w_n)_{n \geq 1}$ is a solution of $Rw = w - \mathbf{1}_{n=0}$ and the result follows. □

• $A.5$ PROOF of LEMMA 6.

We start by a preliminary lemma.

Lemma 24. *Assume* $\lim_{n \to \infty} b_n = 1$. *Then for both the free and zero boundary conditions the Gibbs potential defined in* (2.1), (2.3) *and* (3.1) *is nonpositive.*

Proof: It is enough to prove the result in the case of the bridge since the partition function in the case of free boundary condition is larger. We denote by (B_n) the standard random walk reflected at zero. It is easy to verify that in the case of zero boundary condition

$$Z_{2N} = \mathbf{P}(B_{2N} = 0) \cdot \mathbf{E}\left(e^{\sum_{j=0}^{2N} \log b_{B_j}} \mid B_{2N=0}\right).$$

Using Jensen's inequality we get

$$Z_{2N} \geq \mathbf{P}(B_{2N} = 0) e^{\sum_{j=0}^{2N} \mathbf{E}(\log\, b_{B_j} \mid B_{2N=0})}.$$

By the Markov property and symmetry, we have

$$\mathbf{E}(\log b_{B_j} \mid B_{2N=0}) = \sum_p \mathbf{P}\left(B_j = p\right) \mathbf{P}(B_{2N-j} = p) \log\, b_p.$$

Let $\epsilon > 0$ be fixed and let $K = K(\epsilon)$ be such that

$$\sup_{p \geq K} |\log\, b_p| \leq \epsilon.$$

We have

$$\mathbf{E}(\log\, b_{B_j} \mid B_{2N=0}) = \sum_{p=0}^{K} \mathbf{P}\left(B_j = p\right) \mathbf{P}(B_{2N-j} = p) \log\, b_p + R_\epsilon$$

with

$$|R_\epsilon| \leq \epsilon.$$

The result follows from the well known result that for any fixed K

$$\lim_{j \to \infty} \mathbf{P}(B_j \leq K) = 0. \ \square$$

We now give the proof of Lemma 6. The existence and uniqueness of $\rho(b_0)$ follow from the results of Lemma 3. Note also that $\rho(b_0) > \rho_*(R) \geq 1$ (see Theorem 2). In order to finish the proof of (i), according to formula (2.9) or (2.12) we only need to prove that

$$\lim_{N \to \infty} \frac{1}{2N} \log \mathbf{P}^{\mathrm{RW}}(X_{2N} = 0) = 0.$$

For $b_0 < b_0^c$, we have a positive solution of $Q_{b_0} v = \rho(b_0) v$, unique modulo a multiplicative constant, given by

$$v_1 = 2\rho(b_0)\sqrt{b_0 b_1}\, v_0$$

and for $p > 1$

$$v_p = \frac{v_1 \mathfrak{w}_p}{4\rho(b_0) b_0 b_1}$$

which can also be written for any $p \geq 1$

$$v_p = \frac{v_0 \mathfrak{w}_p}{2\sqrt{b_0 b_1}}.$$

It follows from Lemma 18 that

$$\lim_{p \to \infty} \frac{v_{p+1}}{v_p} = x_-.$$

Therefore using formula (2.5), (recall that $\exp V(n) = b_n$ and $\exp(-U(n)/2) = v_n$), we get, with $\rho = \rho(b_0)$

$$p_n = \mathbf{P}^{\mathrm{RW}}\left(X_1 = n+1 \mid X_0 = n\right) = \rho^{-1} e^{-\log 2 - V(n)/2 - V(n+1)/2 - U(n+1)/2 + U(n)/2}$$
$$= \frac{1}{2\rho\sqrt{b_n b_{n+1}}} \frac{v_{n+1}}{v_n}.$$

This implies since $\rho > 1$ (see equation (8.2))

$$\lim_{n\to\infty} p_n = \frac{x_-(\rho)}{2\,\rho} < 1/2.$$

It follows (by positive recurrence of the corresponding RW) that

$$\inf_{N>0} \mathbf{P}^{\mathrm{RW}}\left(X_{2N} = 0\right) > 0,$$

hence for $b_0 < b_0^c$ the Gibbs potential is equal to $-\log \rho(b_0)$.

We now consider the case $b_0 > b_0^c$. It is easy to verify that the Gibbs potential is not decreasing in b_0. Moreover, since $b_0^c < \infty$ it follows that

$$\lim_{b_0 \nearrow b_0^c} -\log \rho(b_0) \geq 0.$$

By Lemma 24 we have

$$0 \leq \lim_{b_0 \nearrow b_0^c} -\log \rho(b_0) \leq \Phi\left((b_n)\right) \leq 0. \ \Box$$

• $A.6$ PROOF of PROPOSITION 7.

The positive recurrence of the random walk was just proved above. In the sequel we will need the following mixing results valid for any $p \in \mathbb{Z}_+$, under the positive recurrence of X_n (see Appendix $A.7$):

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{RW}}\left(X_{2N} = 0 \mid X_0 = 2p\right) = 2\nu_0, \tag{8.8}$$
$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{RW}}\left(X_{2N+1} = 0 \mid X_0 = 2p+1\right) = 2\nu_0.$$

We start with a preliminary lemma

Lemma 25. *Assume $b_0 < b_0^c$. Then*

$$0 < \lim_{K\to\infty} \sum_p \mathbf{P}^{\mathrm{RW}}(X_{2K} = p \mid X_0 = 0) e^{\frac{1}{2}U(p) - \frac{1}{2}V(p)} < \infty,$$

and

$$0 < \lim_{K\to\infty} \sum_p \mathbf{P}^{\mathrm{RW}}(X_{2K+1} = p \mid X_0 = 0) e^{\frac{1}{2}U(p) - \frac{1}{2}V(p)} < \infty.$$

The two limits may be different.

Proof: As for the proof of (2.11), using detailed balance we get

$$\sum_p \mathbf{P}^{\mathrm{RW}}(X_N = p \mid X_0 = 0) e^{\frac{1}{2}U(p) - \frac{1}{2}V(p)} = e^{U(0)} \sum_p \mathbf{P}^{\mathrm{RW}}(X_N = 0 \mid X_0 = p) e^{-\frac{1}{2}U(p) - \frac{1}{2}V(p)}.$$

For N even, the sum only runs over the even p's while for N odd the sum only runs over the odd p's. Since $b_0 < b_0^c$ we have by Lemma 16 that $\exp(-U(p)/2) = v_p = v_1 \mathfrak{w}_p(\rho)/(4\rho b_0 b_1)$ behaves like x_-^p. The result follows from (8.8) and Lebesgue's dominated convergence theorem. $\Box$

Lemma 26. *Assume the mixing condition* (8.8)*. Then the two limits*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{X_1,\ldots,X_N}\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_N\right)\sum_{l=1}^{N}\mathbf{1}_{X_l=0}$$

and

$$\lim_{N\to\infty}\frac{1}{2N-1}\sum_{X_1,\ldots,X_{2N-1}}\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_{2N-1}\mid X_{2N}=0\right)\sum_{l=1}^{2N-1}\mathbf{1}_{X_l=0}$$

exist and are equal to ν_0

Proof: In the first case (free boundary condition) we have

$$\sum_{X_1,\ldots,X_N}\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_N\right)\sum_{l=1}^{N}\mathbf{1}_{X_l=0}$$

$$=\sum_{l=1}^{N}\sum_{X_1,\ldots,X_{l-1},X_{l+1},\ldots,X_N}\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_N\right).$$

For $l<N$ we have using equality (2.7), the Markov property and equality (2.10)

$$\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_N\right)$$
$$=Z_N^{-1}\rho^N\mathbf{P}^{\mathrm{RW}}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_N\right)e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}$$
$$=Z_N^{-1}\rho^N\mathbf{P}^{\mathrm{RW}}\left(X_1,\ldots,X_{l-1},X_l=0\right)\mathbf{P}^{\mathrm{RW}}\left(X_{l+1},\ldots,X_N\right)e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}.$$

This implies

$$\sum_{X_1,\ldots,X_{l-1},X_{l+1},\ldots,X_N}^{\mathrm{SOS}}\mathbf{P}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_N\right)$$
$$=Z_N^{-1}\rho^N\mathbf{P}^{\mathrm{RW}}\left(X_l=0\right)\sum_{X_{N-l}}\mathbf{P}^{\mathrm{RW}}\left(X_{N-l}\right)e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_{N-l})-\frac{1}{2}V(X_{N-l})}$$
$$=\frac{\mathbf{P}^{\mathrm{RW}}\left(X_l=0\right)\sum_{X_{N-l}}\mathbf{P}^{\mathrm{RW}}\left(X_{N-l}\right)e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_{N-l})-\frac{1}{2}V(X_{N-l})}}{\sum_{X_N}\mathbf{P}^{\mathrm{RW}}(X_N)e^{-\frac{1}{2}U(0)-\frac{1}{2}V(0)+\frac{1}{2}U(X_N)-\frac{1}{2}V(X_N)}}$$

Using Lemma 25, the mixing condition (8.8) and Lebesgue's dominated convergence theorem the result follows. Note that $\mathbf{P}^{\mathrm{RW}}\left(X_l=0\right)=0$ if l is odd.

In the case of the bridge (zero boundary condition) we get

$$\sum_{X_1,\ldots,X_{2N-1}}\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_{2N-1}\mid X_{2N}=0\right)\sum_{l=1}^{2N-1}\mathbf{1}_{X_l=0}$$

$$=\sum_{l=1}^{2N-1}\sum_{\substack{x_1,\ldots,x_{l-1},\\ X_{l+1},\ldots,X_{2N-1}}}^{\mathrm{SOS}}\mathbf{P}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_{2N-1}\mid X_{2N}=0\right).$$

For $l\le 2N-1$ we have using equality (2.7), the Markov property and equality (2.10)

$$\mathbf{P}^{\mathrm{SOS}}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_{2N-1}\mid X_{2N}=0\right)$$
$$=\mathbf{P}^{\mathrm{RW}}\left(X_1,\ldots,X_{l-1},0,X_{l+1},\ldots,X_{2N-1}\mid X_{2N}=0\right)$$
$$=\mathbf{P}^{\mathrm{RW}}\left(X_1,\ldots,X_{l-1},X_l=0\right)\mathbf{P}^{\mathrm{RW}}\left(X_{l+1},\ldots,X_{2N-1}\mid X_{2N}=0\right).$$

This implies

$$\sum_{\substack{X_1,\dots,X_{l-1},\\ X_{l+1},\dots,X_{2N-1}}} \mathbf{P}^{\mathrm{SOS}}\left(X_1,\dots,X_{l-1},0,X_{l+1},\dots,X_{2N-1} \mid X_{2N}=0\right) = \mathbf{P}^{\mathrm{RW}}\left(X_l = 0 \mid X_{2N}=0\right).$$

The result follows from the mixing condition (8.8). □

Proposition 27. *Assume* $\lim_{n\to\infty} b_n = 1$.

(i) *Then for any* $b_0 > 0$, *the operator* Q_{b_0} *has an essential spectral radius in* $l^p(\mathbb{Z}_+)$ *equal to one, for any* $1 \le p \le \infty$.

(ii) *All its eigenvalues are simple.*

(iii) *The operator* Q_{b_0} *is self adjoint in* $l^2(\mathbb{Z}_+)$.

(iv) *For* $b_0 < b_0^c$, $\rho(b_0) := \rho_*(Q_{b_0})$ *is the largest eigenvalue of* Q_{b_0} *in* $l^2(\mathbb{Z}_+)$.

(v) *For* $b_0 < b_0^c$, $\rho(b_0)$ *is real analytic in* b_0, *and*

$$\partial_{b_0}\rho(b_0) = -\frac{\rho(b_0)v_0^2}{b_0\sum_{j=0}^{\infty} v_j^2}.$$

Proof:

(i) Given $\epsilon > 0$, let N be such that

$$\sup_{n\ge N} |1-b_n| \le \epsilon.$$

Define the infinite matrix $Q_{b_0}^{(N)}$ by

$$Q_{b_0}^{(N)}(i,j) = Q_{b_0}(i,j)\cdot \mathbf{1}_{i\ge N}\cdot \mathbf{1}_{j\ge N}.$$

It is easy to verify that

$$\left\| Q_{b_0}^{(N)} \right\|_{l^p(\mathbb{Z}_+)} \le 1+\epsilon.$$

Therefore its essential spectral radius is at most $1+\epsilon$. Since $Q_{b_0} - Q_{b_0}^{(N)}$ is a finite rank operator, we conclude by Nussbaum's Theorem (see [18]) that the essential spectral radius of Q_{b_0} is also at most $1+\epsilon$. Since this is true for any $\epsilon > 0$, the result follows.

(ii) This follows from the fact that if $Q_{b_0}v = \rho\, v$, then v_2 is determined by v_1 and then recursively for all v_n with $n \ge 2$ since the equation is a recursion of order two.

(iii) Q_{b_0} is real-symmetric and bounded.

(iv) Assume the largest eigenvalue in $\ell^2(\mathbb{Z}_+)$ is $\tilde\rho > \rho(b_0)$. Let $\tilde v$ denote the corresponding normalized eigenvector, and denote by v the positive eigenvector corresponding to ρ_{b_0}. We first claim that $v_n > 0$ for all n. Indeed if $v_n = 0$ for some $n \ge 1$, we have (by the positivity of the sequence and the equation $(Q_{b_0}v)_n = 0$) $v_{n\pm1} = 0$. This implies $v = 0$. If $v_0 = 0$, it follows that $v_1 = 0$, and this implies by a recursive argument $v = 0$. For any positive $h \in \ell^2(\mathbb{Z}_+)$, and for any integer n, $Q^n h$ is a positive sequence, moreover

$$\lim_{n\to\infty} \tilde\rho^{-n} Q^n h = \langle \tilde v\,,\, h\rangle \cdot \tilde v$$

where $\langle v\,,\, h\rangle$ denotes the scalar product in $\ell^2(\mathbb{Z}_+)$. Since any complex sequence can be obtained by a linear combination of at most four positive sequences, we conclude that there exists a positive $h \in \ell^2(\mathbb{Z}_+)$ such that $\langle \tilde v\,,\, h\rangle \neq 0$. This implies that all the $\tilde{v}_n$ have the same sign. However this contradicts the well known fact that if $\tilde{\rho} \neq \rho(b_0)$, then $\langle \tilde{v}\,,\, v\rangle = 0$.

(v) Follows from analytic perturbation theory of a simple eigenvalue, see for example [13].

The formula displayed in (v) can also be written as:

$$-\frac{\partial \log \rho(b_0)}{\partial \log(b_0)} = \frac{\partial \Phi((b_n))}{\partial \log(b_0)} = \frac{v_0^2}{\sum_{j=0}^{\infty} v_j^2},$$

when $b_0 < b_0^c$, in agreement with Lemma 26.

We now give the proof of Proposition 7.

(i) Follows from Lemma 26.

(ii) Follows from Lemma 26 and the following computation. We have

$$v_1 = 2\rho\sqrt{b_0\, b_1} v_0$$

and for $p \geq 1$

$$v_p = \frac{v_1}{4\,\rho\, b_0\, b_1}\mathfrak{w}_p(\rho) = \frac{v_0}{2\sqrt{b_0\, b_1}}\,\mathfrak{w}_p(\rho).$$

Hence

$$\frac{v_0^2}{\sum_{p=0}^{\infty} v_p^2} = \frac{v_0^2}{v_0^2 + \sum_{p=1}^{\infty} v_p^2} = \frac{1}{1 + (4\, b_0\, b_1)^{-1}\sum_{p=1}^{\infty} \mathfrak{w}_p(\rho)^2}.$$

(iii) Follows by a standard convexity argument.

(iv) Follows from Proposition 27 and Lemma 6.

• $A.7$ ANOTHER APPROACH to Z_N and a PROOF of (4.1).

We now come back to the factor 2 in equation (8.8) and make some more comments.

One can express the partition function Z_N in terms of the infinite matrix Q_{b_0}. It is easy to verify that in the case of the bridge

$$Z_N = e^{-V(0)}\langle 0 \,|\, Q_{b_0}^N\,|0\rangle$$

where $|0\rangle$ is the sequence

$$|0\rangle_n = \mathbf{1}_{n=0}.$$

In the free boundary case one has

$$Z_N = e^{-V(0)/2}\langle e^{-V/2}\,|\, Q_{b_0}^N\,|0\rangle$$

where $|e^{-V/2}\rangle$ is the sequence

$$|e^{-V/2}\rangle_n = e^{-V(n)/2}.$$

These expressions lead to another proof of part (i) in Lemma 6 using the spectral theory of Q_{b_0} (in $l^2(\mathbb{Z}_+)$ and $l^1(\mathbb{Z}_+)$ respectively).

Let S denote the involution acting on sequences by

$$(Sh)_n = (-1)^n h_n$$

It is easy to verify that

$$SQ_{b_0}S = -Q_{b_0}.$$

This implies that the spectrum of Q_{b_0} is invariant by multiplication by -1. In particular, $-\rho(b_0)$ is also an eigenvalue with eigenvector Sv. Since $S|0\rangle = |0\rangle$, we find that for any integer N

$$\langle 0 \,|\, Q_{b_0}^{2N+1} \,|0\rangle = 0$$

and

$$\lim_{N\to\infty} \rho(b_0)^{-2N} \ \langle 0 \,|\, Q_{b_0}^{2N} \,|0\rangle = \frac{2\ v_0^2}{\|v\|^2_{\ell^2(\mathbb{Z}_+)}}.$$

The same result holds for the Markov operator P_{b_0} associated to the walk which is conjugated to Q_{b_0}, namely with obvious (Hadamard) notation

$$P_{b_0} h = \frac{1}{\rho(b_0) v} Q_{b_0}(v\, h).$$

Since

$$P_{b_0}^{2N}\,(0, 2p) = \mathbf{P}^{\mathrm{RW}}(X_{2N} = 2p) = \frac{v_{2p}}{v_0 \rho(b_0)^{2N}} \cdot \langle 0 \,|\, Q_{b_0}^{2N} \,|2p\rangle$$

we get

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{RW}}(X_{2N} = 2p) = \frac{2 v_{2p}^2}{\|v\|^2_{\ell^2(\mathbb{Z}_+)}}.$$

which implies the first statement in (8.8). One proves similarly that

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{RW}}(X_{2N+1} = 2p+1) = \frac{2 v_{2p+1}^2}{\|v\|^2_{\ell^2(\mathbb{Z}_+)}}.$$

Finally, since for all N

$$\sum_{p=0}^{\infty} \mathbf{P}^{\mathrm{RW}}(X_{2N} = 2p) = 1$$

we get

$$\sum_{p=0}^{\infty} v_{2p}^2 = \frac{1}{2} \sum_{p=0}^{\infty} v_p^2.$$

• $A.8$ PROOF of THEOREM 8.

- Proof of Theorem 8.(i)**.**

It follows from Lemma 18 that for large p, v_p which is proportional to $\mathfrak{w}_p$ behaves like $\sim x_-^p$. Since this sequence belongs to l^2 and $v_0 > 0$, from (4.1), we get $m > 0$.

- Proof of Theorem 8.(ii).

Let $b_0 > \beta > b_0^c$, and denote the number of returns to zeros between 1 and K by

$$\mathfrak{N}_K = \sum_{j=1}^{K} \mathbf{1}_{X_j = 1}.$$

By Jensen's inequality

$$Z_{2N}(\beta) = Z_{2N}(b_0) \langle e^{\mathfrak{N}_{2N-1}(\log b_0 - \log \beta)} \rangle_{b_0, 2N} \geq Z_{2N}(b_0) e^{(\log b_0 - \log \beta) < \mathfrak{N}_{2N-1} >_{b_0, 2N}}$$

and

$$\limsup_{N\to\infty} \frac{< \mathfrak{N}_{2N-1} >_{b_0,\,2N}}{2N} \leq \frac{1}{\log b_0 - \log \beta} \left(\lim_{N\to\infty} \frac{\log Z_{2N}(\beta)}{2N} - \lim_{N\to\infty} \frac{\log Z_{2N}(b_0)}{2N} \right) = 0,$$

by Lemma 6.(ii). The result follows since $\mathfrak{N}_N \geq 0$.

- Proof of Theorem 8.(iii)**.**

In this case using Corollary 23 we have

$$\sum_p \mathfrak{w}_p(1)^2 = \infty.$$

For any $K > 0$, let $N(K) > 1$ be an integer such that

$$\sum_{p=1}^{N(K)} \mathfrak{w}_p(1)^2 > K.$$

By Lemma 3.(v) we have

$$\lim_{\rho \nearrow 1} \sum_p \mathfrak{w}_p(\rho)^2 \geq \lim_{\rho \nearrow 1} \sum_{p=1}^{N(K)} \mathfrak{w}_p(\rho)^2 > K.$$

Since this holds for any K we conclude that

$$\lim_{\rho \nearrow 1} \sum_p \mathfrak{w}_p(\rho)^2 = \infty.$$

The result follows from formula 4.1. □

- Proof of Theorem 8.(iv)**.**

From Corollary 23 we have

$$\sum_p \mathfrak{w}_p(1)^2 < \infty$$

if and only if $\alpha_- < -1/2$ which gives $w < -3/8$.

If $w < -3/8$, we have by Proposition 19 for any $\rho \geq 1$

$$\sum_p \mathfrak{w}_p(\rho)^2 \leq \sum_p \mathfrak{w}_p(1)^2 < \infty$$

and the result follows from formula (4.1). □

• $A.9$ PROOF of THEOREM 11.

We start with several preliminary lemmas.

Lemma 28. *For u and s real*

$$z\frac{(u+1/2)(-u-s)}{(2u+s)(2u+s+1)}F(u+1, u+3/2; 2u+2+s; z)$$
$$= F(u, u+1/2; 2u+s; z) - F(u+1/2, u+1; 2u+1+s; z).$$

Proof: We first observe that

$$F(u+1/2, u+1; 2u+1+s; z) = F(u+1, u+1/2; 2u+1+s; z).$$

Therefore we need to prove that

$$z\frac{(u+1/2)(-u-s)}{(2u+s)(2u+s+1)}F(u+1, u+3/2; 2u+2+s; z)$$
$$= F(u, u+1/2; 2u+s; z) - F(u+1, u+1/2; 2u+1+s; z)$$

This follows from formula **9.137.16** in [9] by taking

$$\alpha = u,\ \beta = u + 1/2,\ \gamma = 2u + s.\ \square$$

Lemma 29. *Let a real, define for all $n \geq -1$*

$$P_n(z) = F(a + n/2, a + (n+1)/2; 2a + n + s; z).$$

Then

$$P_n(z) = \frac{4}{z}\frac{(2a+n+s-2)(2a+n+s-1)}{(2a+n-1)(2a+n-2+2s)}\left(P_{n-1}(z) - P_{n-2}(z)\right).$$

Proof: Apply Lemma 28 with $u = a + n/2 - 1$. $\square$

Lemma 30. *For $n \geq 1$, and $\rho \geq 1$, define*

$$w_n = C(2\rho)^{-n} P_{n-1}(\rho^{-2})\sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)\Gamma(2a+n-1)\Gamma(2s+2a+n-2)}{\Gamma(s+n-2+2a)\Gamma(s+n-1+2a)\Gamma(2a)\Gamma(2s+2a-1)}}$$

where C is a constant, and the $P_n(\cdot)$ are defined in Lemma 29. Then for any $n > 1$ we have

$$(Rw)_n = \rho w_n.$$

Moreover, if

$$C = \frac{2}{F(a-1/2, a; 2a-1+s; \rho^{-2})},$$

then

$$(Rw)_1 = \rho w_1 - 1.$$

Proof: For $n \geq 0$, denote by T_n the product

$$T_n = \sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)\Gamma(2a+n-1)\Gamma(2s+2a+n-2)}{\Gamma(s+n-2+2a)\Gamma(s+n-1+2a)\Gamma(2a)\Gamma(2s+2a-1)}}.$$

For $n \geq 0$ we have

$$T_{n+1} = \left(\frac{(s+n-2+2a)(s+n-1+2a)}{(2a+n-1)(2s+2a+n-2)}\right)^{-1/2} T_n. \tag{8.9}$$

Observe that for $n \geq 1$

$$b_n b_{n+1} = \frac{(s+n-2+2a)(s+n+2a-1)}{4(a+n/2-1/2)(s+a+n/2-1)} = \frac{(s+n-2+2a)(s+n-1+2a)}{(2a+n-1)(2s+2a+n-2)}. \tag{8.10}$$

Since

$$w_n = C(2\rho)^{-n} P_{n-1}(\rho^{-2}) T_n,$$

we have for any $n \geq 1$ using Lemma 29 and equation (8.9)

$$\frac{w_{n+1}}{2\sqrt{b_n b_{n+1}}} - \rho w_n = C(2\rho)^{-n-1} P_n \frac{T_{n+1}}{2\sqrt{b_n\, b_{n+1}}} - \rho C(2\rho)^{-n} P_{n-1} T_n$$

$$= C\rho(2\rho)^{-n} T_n \left(\frac{P_n}{4\rho^2 b_n b_{n+1}} - P_{n-1}\right) = -C\rho(2\rho)^{-n} T_n P_{n-2}$$

$$= -\frac{1}{2}\left(\frac{(s+n-3+2a)(s+n-2+2a)}{(2a+n-2)(2s+2a+n-3)}\right)^{-1/2} C(2\rho)^{-n+1} T_{n-1} P_{n-2}.$$

The right hand side is equal to $w_{n-1}/(2\sqrt{b_n b_{n-1}}$ if $n > 1$ from (8.10) and the definition of w_{n-1}.

We now choose

$$C = 2\left(\frac{(s-2+2a)(s-1+2a)}{(2a-1)(2s+2a-2)}\right)^{1/2} \frac{1}{T_0 P_{-1}(\rho^{-2})} = \frac{2}{F(a-1/2, a; 2a-1+s; \rho^{-2})}$$

since

$$T_0^2 = \frac{\Gamma(s-1+2a)\Gamma(s+2a)\Gamma(2a-1)\Gamma(2s+2a-2)}{\Gamma(s-2+2a)\Gamma(s-1+2a)\Gamma(2a)\Gamma(2s+2a-1)}$$
$$= \frac{(s-2+2a)(s-1+2a)}{(2a-1)(2s+2a-2)}.$$

With this choice of C, we get

$$(Rw)_1 = \rho w_1 - 1. \ \square$$

For large n, using Watson's asymptotics (see [7] for example), or directly by steepest descent form the integral representation **2.12.1** in [7], one gets that for $\rho > 1$

$$w_n \sim (\rho - \sqrt{\rho^2 - 1})^n.$$

From Lemma 16 since $w > 0$, we get $\mathfrak{w} = w > 0$ for $\rho > 1$. In particular this implies that $\rho_*(R) = 1$. This proves (i) and (ii) of Theorem 11.

For $\rho = 1$, using Lemma 30, we have for any $p \geq 1$

$$\mathfrak{w}_p(1) = 2(2)^{-p} \frac{F(a+(p-1)/2, a+p/2; 2a+p+s-1; 1)}{F(a-1/2, a; 2a+s-1; 1)} \times$$
$$\sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)\Gamma(2a+p-1)\Gamma(2s+2a+p-2)}{\Gamma(s+p-2+2a)\Gamma(s+p-1+2a)\Gamma(2a)\Gamma(2s+2a-1)}}$$

$$= 2(2)^{-p} \frac{\Gamma(2a+p+s-1)\Gamma(s-1/2)}{\Gamma(a+s+p/2-1/2)\Gamma(a+s+p/2-1)} \frac{\Gamma(a+s-1/2)\Gamma(a+s-1)}{\Gamma(2a+s-1)\Gamma(s-1/2)} \times$$
$$\sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)}{\Gamma(2a)\Gamma(2s+2a-1)}} \sqrt{\frac{\Gamma(2a+p-1)\Gamma(2s+2a+p-2)}{\Gamma(s+p-2+2a)\Gamma(s+p-1+2a)}}.$$

By the duplication formula for the Γ function (see for example [7])

$$\mathfrak{w}_p(1) = 2(2)^{-p} 2^p \frac{\Gamma(2a+p+s-1)}{\Gamma(2a+2s+p-2)} \frac{\Gamma(2a+2s-2)}{\Gamma(2a+s-1)} \times$$
$$\sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)}{\Gamma(2a)\Gamma(2s+2a-1)}} \sqrt{\frac{\Gamma(2a+p-1)\Gamma(2s+2a+p-2)}{\Gamma(s+p-2+2a)\Gamma(s+p-1+2a)}}$$
$$= 2\frac{\Gamma(2a+2s-2)}{\Gamma(2a+s-1)} \sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)}{\Gamma(2a)\Gamma(2s+2a-1)}} \times$$
$$\frac{\Gamma(2a+p+s-1)}{\Gamma(2a+2s+p-2)} \sqrt{\frac{\Gamma(2a+p-1)\Gamma(2s+2a+p-2)}{\Gamma(s+p-2+2a)\Gamma(s+p-1+2a)}}$$
$$= 2\frac{2a+s-1}{2a+2s-2} \frac{\Gamma(2a+2s-1)}{\Gamma(2a+s)} \sqrt{\frac{\Gamma(s-1+2a)\Gamma(s+2a)}{\Gamma(2a)\Gamma(2s+2a-1)}} \times$$

$$\frac{\Gamma(2a+p+s-1)}{\Gamma(2a+2s+p-2)}\sqrt{\frac{\Gamma(2a+p-1)\Gamma(2s+2a+p-2)}{\Gamma(s+p-2+2a)\Gamma(s+p-1+2a)}}$$

$$= 2\frac{2a+s-1}{2a+2s-2}\sqrt{\frac{\Gamma(s-1+2a)\Gamma(2a+2s-1)}{\Gamma(2a)\Gamma(s+2a)}}\sqrt{\frac{\Gamma(2a+p-1)\Gamma(2a+p+s-1)}{\Gamma(s+p-2+2a)\Gamma(2a+2s+p-2)}}$$

$$= \frac{1}{a+s-1}\sqrt{(2a+p+s-2)(2a+s-1)}\sqrt{\frac{\Gamma(2a+2s-1)}{\Gamma(2a)}}\sqrt{\frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)}}.$$

This gives the critical $\mathfrak{w}_p(1)$ for all $p \geq 1$, with, by Stirling's formula

$$\mathfrak{w}_p(1) = \mathcal{O}(1) \cdot p^{(1-s)} \text{ for large } p.$$

We also have

$$b_1 = \frac{(s+2a-1)\Gamma(a)\Gamma(s+a-1/2)}{2\Gamma(a+1/2)\Gamma(s+a)},$$

hence from the expression of $\mathfrak{w}_1(1)$

$$b_0^c = \frac{\mathfrak{w}_1(1)}{4b_1} = \frac{1}{2}\frac{\Gamma(a+s-1)\Gamma(a+1/2)}{\Gamma(a)\Gamma(s+a-1/2)},$$

proving (iii) of Theorem 11.

For $b_0 < b_0^c$, the equation for $\rho(b)$ is given in Lemma 6.(i) and we can replace $\mathfrak{w}_1(\rho)$ by its explicit expression.

• $A.10$ PROOF of PROPOSITION 12.

Recall that for $b_0 \leq b_0^c$, from Proposition 7.(iv) we have

$$m(b_0) = -b_0\frac{\partial_{b_0}\rho(b_0)}{\rho(b_0)}.$$

In order to compute $\partial_{b_0}\rho(b_0)$, we use

$$\mathfrak{w}_1(\rho) = \frac{F(a,a+1/2;2a+s;\rho^{-2})}{\rho F(a-1/2,a;2a+s-1;\rho^{-2})}$$

and we take the derivative with respect to b_0 of the implicit equation for $\rho(b_0)$ given in Lemma 6:

(8.11)

$$F(a,a+1/2;2a+s;\rho^{-2}(b_0)) - 4\rho(b_0)^2 b_0 b_1 F(a-1/2,a;2a+s-1;\rho^{-2}(b_0)) = 0.$$

We get

$$-2F(a+1,a+3/2;2a+s+1;\rho^{-2})\frac{a(a+1/2)}{2a+s}\frac{\partial_{b_0}\rho}{\rho^3} - 8\rho\,(\partial_{b_0}\rho)\,b_1 b_0 F(a-1/2,a;2a+s-1;\rho^{-2})$$
$$-4\rho(b_0)^2 b_1 F(a-1/2,a;2a+s-1;\rho^{-2}(b_0))$$
$$+8b_1 b_0 F(a+1/2,a+1;2a+s;\rho^{-2}(b_0))\frac{a(a-1/2)}{2a+s-1}\frac{\partial_{b_0}\rho}{\rho} = 0,$$

which is also

$$-2\frac{F(a+1,a+3/2;2a+s+1;\rho^{-2})}{F(a-1/2,a;2a+s-1;\rho^{-2})}\frac{a(a+1/2)}{2a+s}\frac{\partial_{b_0}\rho}{\rho^3} - 8\rho\,(\partial_{b_0}\rho)\,b_1 b_0$$
$$-4\rho^2 b_1 + 8b_1 b_0\frac{F(a+1/2,a+1;2a+s;\rho^{-2})}{F(a-1/2,a;2a+s-1;\rho^{-2})}\frac{a(a-1/2)}{2a+s-1}\frac{\partial_{b_0}\rho}{\rho} = 0.$$

From (see Proposition 7.(iv))

$$\partial_{b_0}\rho = -\frac{m\rho}{b_0},$$

we get

$$\frac{1}{2\rho^2 b_1}\frac{F(a+1,a+3/2;2a+s+1;\rho^{-2})}{F(a-1/2,a;2a+s-1;\rho^{-2})}\frac{a(a+1/2)}{2a+s}\frac{m}{b_0\rho^2}+2m$$
$$-1-\frac{2}{\rho^2}\frac{F(a+1/2,a+1;2a+s;\rho^{-2})}{F(a-1/2,a;2a+s-1;\rho^{-2})}\frac{a(a-1/2)}{2a+s-1}m=0.$$

We can now use again equation (8.11) to get

$$\frac{2}{\rho^2}\frac{F(a+1,a+3/2;2a+s+1;\rho^{-2})}{F(a,a+1/2;2a+s;\rho^{-2})}\frac{a(a+1/2)}{2a+s}m+2m$$
$$-1-\frac{2}{\rho^2}\frac{F(a+1/2,a+1;2a+s;\rho^{-2})}{F(a-1/2,a;2a+s-1;\rho^{-2})}\frac{a(a-1/2)}{2a+s-1}m=0,$$

and we get the claimed expression (6.1) for m.

We have to estimate the denominator of equation (6.1) for $\rho(b_0)$ near one.

We first consider the case $1/2<s<3/2$.

Recall that if $\gamma>0$ and $\gamma>\alpha+\beta$

$$\lim_{z\nearrow 1}F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

Therefore

$$\lim_{z\nearrow 1}F(a-1/2,a;2\,a+s-1;z)=\frac{\Gamma(2a+s-1)\Gamma(s-1/2)}{\Gamma(a+s-1/2)\Gamma(a+s-1)}$$

(since in that case $\gamma-\beta-\alpha=s-1/2>0$). Similarly

$$\lim_{z\nearrow 1}F(a,a+1/2;2\,a+s;z)=\frac{\Gamma(2a+s)\Gamma(s-1/2)}{\Gamma(a+s)\Gamma(a+s-1/2)}.$$

If $\gamma>0$ and $\gamma<\alpha+\beta$, the limit is infinite. More precisely, it follows from formula **2.10.1** in [7] that if $-1<\gamma-\alpha-\beta<0$

$$F(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\beta-\alpha}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}+\mathcal{O}(1).$$

Therefore for $1/2<s<3/2$ we get

$$F(a+1,a+3/2;2\,a+s+1;z)=(1-z)^{s-3/2}\frac{\Gamma(2\,a+s+1)\Gamma(3/2-s)}{\Gamma(a+1)\Gamma(a+3/2)}+\mathcal{O}(1), \tag{8.12}$$

and

$$F(a+1/2,a+1;2\,a+s;z)=(1-z)^{s-3/2}\frac{\Gamma(2\,a+s)\Gamma(3/2-s)}{\Gamma(a+1/2)\Gamma(a+1)}+\mathcal{O}(1).$$

Therefore, after simple algebraic manipulations, we get

$$\frac{F(a+1,a+3/2;2a+s+1;z)}{F(a,a+1/2;2a+s;z)}\frac{(a+1/2)}{2a+s}-\frac{F(a+1/2,a+1;2a+s;z)}{F(a-1/2,a;2a+s-1;z)}\frac{(a-1/2)}{2a+s-1}$$
$$=(1-z)^{s-3/2}\frac{\Gamma(2\,a+s-1)\Gamma(3/2-s)}{\Gamma(a+1/2)\Gamma(a+1)}(a+s-1)+\mathcal{O}(1).$$

From (6.1), this implies that for $1/2 < s < 3/2$

$$m(b_0) = \left(1-\rho(b_0)^{-2}\right)^{3/2-s} \frac{\Gamma(a+1/2)\,\Gamma(a+1)}{2a(a+s-1)\Gamma(2a+s-1)\Gamma(3/2-s)} \left(1+\mathcal{O}\left(1-\rho(b_0)^{-2}\right)\right)^{3/2-s}.$$

In particular, we see again that for $1/2 < s < 3/2$

$$\lim_{b_0 \nearrow b_0^c} m(b_0) = 0.$$

In order to be able to compute the critical indices, we need to know how $\rho(b_0)-1$ vanishes as a function of $b_0 - b_0^c$ when $b_0 \nearrow b_0^c$.

We have

$$b_0 - b_0^c = \frac{\mathfrak{w}_1(\rho(b_0))}{4\rho(b_0)b_1} - \frac{\mathfrak{w}_1(1)}{4b_1}$$

$$= \frac{1}{4\rho(b_0)^2 b_1} \frac{F(a,a+1/2;2a+s;\rho(b_0)^{-2})}{F(a-1/2,a;2a+s-1;\rho(b_0)^{-2})} - \frac{1}{4b_1}\frac{F(a,a+1/2;2a+s;1)}{F(a-1/2,a;2a+s-1;1)}.$$

For any $0<z<\xi<1$, we can write, using formula **2.8.20** in [7]

$$F(a,a+1/2;2a+s;\xi) - F(a,a+1/2;2a+s;z) = \int_z^\xi \frac{dF(a,a+1/2;2a+s;t)}{dt}dt$$

$$= \frac{a(a+1/2)}{2a+s}\int_z^\xi F(a+1,a+3/2;2a+s+1;t)dt.$$

Using the identity (8.12) this is equal to

$$\frac{a(a+1/2)}{2a+s}\int_z^\xi \left((1-t)^{s-3/2}\frac{\Gamma(2\,a+s+1)\,\Gamma(3/2-s)}{\Gamma(a+1)\Gamma(a+3/2)} + \mathcal{O}(1)\right)dt$$

$$= -\frac{\Gamma(2a+s)\Gamma(3/2-s)}{\Gamma(a)\Gamma(a+1/2)}\frac{1}{s-1/2}\left((1-\xi)^{s-1/2}-(1-z)^{s-1/2}\right) + \mathcal{O}(\xi-z).$$

We can now let ξ tend to one and get

$$F(a,a+1/2;2a+s;1) - F(a,a+1/2;2a+s;z)$$

$$= \frac{\Gamma(2a+s)\,\Gamma(3/2-s)}{\Gamma(a)\Gamma(a+1/2)}\frac{1}{s-1/2}(1-z)^{s-1/2} + \mathcal{O}(1-z).$$

In other words

$$F(a,a+1/2;2a+s;z)$$

$$= \frac{\Gamma(2a+s)\Gamma(s-1/2)}{\Gamma(a+s)\Gamma(a+s-1/2)} - \frac{\Gamma(2\,a+s)\Gamma(3/2-s)}{\Gamma(a)\Gamma(a+1/2)}\frac{1}{s-1/2}(1-z)^{s-1/2} + \mathcal{O}(1-z)$$

$$= \frac{\Gamma(2a+s)\Gamma(s-1/2)}{\Gamma(a+s)\Gamma(a+s-1/2)}\left(1 - \frac{\Gamma(a+s)\Gamma(a+s-1/2)}{\Gamma(a)\Gamma(a+1/2)}\frac{1}{(s-1/2)^2}(1-z)^{s-1/2}\right) + \mathcal{O}(1-z).$$

Replacing a by $a-1/2$ we get

$$F(a-1/2,a;2a+s-1;1) - F(a-1/2,a;2a+s-1;z)$$

$$= \frac{\Gamma(2\,a+s-1)\Gamma(3/2-s)}{\Gamma(a-1/2)\Gamma(a)}\frac{1}{s-1/2}(1-z)^{s-1/2} + \mathcal{O}(1-z).$$

In other words

$$F(a-1/2,a;2a+s-1;z)$$

$$= \frac{\Gamma(2a+s-1)\Gamma(s-1/2)}{\Gamma(a+s-1/2)\Gamma(a+s-1)} - \frac{\Gamma(2\,a+s-1)\Gamma(3/2-s)}{\Gamma(a-1/2)\Gamma(a)}\frac{1}{s-1/2}(1-z)^{s-1/2} + \mathcal{O}(1-z)$$

$$= \frac{\Gamma(2a+s-1)\Gamma(s-1/2)}{\Gamma(a+s-1/2)\Gamma(a+s-1)}\left(1 - \frac{\Gamma(a+s-1/2)\Gamma(a+s-1)}{\Gamma(a-1/2)\Gamma(a)}\frac{1}{(s-1/2)^2}(1-z)^{s-1/2}\right) + \mathcal{O}(1-z)$$

We obtain the following estimate of $b_0 - b_0^c$ (near $z = \rho(b_0)^{-2} = 1$)

$$\frac{1}{4b_1}\left(\frac{F(a,a+1/2;2a+s;\rho(b_0)^{-2})}{F(a-1/2,a;2a+s-1;\rho(b_0)^{-2})} - \frac{F(a,a+1/2;2a+s;1)}{F(a-1/2,a;2a+s-1;1)}\right) + \mathcal{O}(1-z)$$
$$= \frac{1}{4b_1}\frac{\Gamma(2a+s)\Gamma(s-1/2)}{\Gamma(a+s)\Gamma(a+s-1/2)}\frac{\Gamma(a+s-1/2)\Gamma(a+s-1)}{\Gamma(2a+s-1)\Gamma(s-1/2)}\frac{1}{(s-1/2)^2}(1-z)^{s-1/2}\times$$
$$\left(\frac{\Gamma(a+s-1/2)\Gamma(a+s-1)}{\Gamma(a-1/2)\Gamma(a)} - \frac{\Gamma(a+s)\Gamma(a+s-1/2)}{\Gamma(a)\Gamma(a+1/2)}\right)$$
$$= \frac{1}{4b_1}\frac{2a+s-1}{a+s-1}\frac{1}{(s-1/2)^2}(1-z)^{s-1/2}\frac{\Gamma(a+s-1/2)\,\Gamma(a+s-1)}{\Gamma(a-1/2)\,\Gamma(a)}\left(1-\frac{a+s-1}{a-1/2}\right)$$
$$= -\frac{1}{4b_1}\frac{2a+s-1}{a+s-1}\frac{1}{(s-1/2)(a-1/2)}(1-z)^{s-1/2}\frac{\Gamma(a+s-1/2)\Gamma(a+s-1)}{\Gamma(a-1/2)\Gamma(a)}.$$

Therefore, for $b < b_0^c$ and $1/2 < s < 3/2$

$$m(b_0) = C(b_0^c - b_0)^{(3/2-s)/(s-1/2)}(1 + o(1))$$

where C is a positive constant that can be explicitly computed.

Finally for $s > 3/2$

$$\lim_{z\nearrow 1}\frac{F(a+1,a+3/2;2a+s+1;z)}{F(a,a+1/2;2a+s;z)} = \frac{\Gamma(2a+s+1)\Gamma(s-3/2)}{\Gamma(a+s)\Gamma(a+s-1/2)}\frac{\Gamma(a+s)\Gamma(a+s-1/2)}{\Gamma(2a+s)\Gamma(s-1/2)} = \frac{2a+s}{s-3/2}.$$

$$\lim_{z\nearrow 1}\frac{F(a+1/2,a+1;2a+s;z)}{F(a-1/2,a;2a+s-1;z)} = \frac{\Gamma(2a+s)\Gamma(s-3/2)}{\Gamma(a+s-1/2)\Gamma(a+s-1)}\frac{\Gamma(a+s-1/2)\Gamma(a+s-1)}{\Gamma(2a+s-1)\Gamma(s-1/2)} = \frac{2a+s-1}{s-3/2}.$$

We get for $s > 3/2$

$$\lim_{b_0\nearrow b_0^c} m(b_0) = \frac{1}{2+2a\left(\frac{2a+s}{s-3/2}\frac{(a+1/2)}{2a+s} - \frac{2a+s-1}{s-3/2}\frac{(a-1/2)}{2a+s-1}\right)} = \frac{1}{2+\frac{2a}{(s-3/2)}}.$$

This completes the proof of Proposition 12. □

Remark: The form of the critical index can be guessed by the following argument. We have the relation

$$\frac{\mathfrak{w}_1(\rho)}{4\rho b_1} = b_0,$$

and taking the partial derivative with respect to ρ we get

$$\frac{\mathfrak{w}_1'(\rho)}{4\rho b_1} - \frac{\mathfrak{w}_1(\rho)}{4\rho^2 b_1} = \frac{db_0}{d\rho}.$$

Assume we know (as we saw before) that

$$\frac{1}{m} = \frac{db_0}{d\rho} = \mathcal{O}(1)\cdot(\rho-1)^{\alpha},$$

then

$$b_0^c - b_0 = \int_{\rho(b_0)}^1 \frac{db_0}{d\rho}d\rho = \int_{\rho(b_0)}^1\left(\frac{\mathfrak{w}_1'(\rho)}{4\rho b_1} - \frac{\mathfrak{w}_1(\rho)}{4\rho^2 b_1}\right)d\rho = \mathcal{O}(1)\cdot(\rho-1)^{\alpha-1},$$

and we get

$$m = \mathcal{O}(1)\cdot(\rho-1)^{-\alpha/(\alpha-1)},$$

which is our result if we replace α by $3/2 - s$.

• A.11 COMPUTATIONS OF SECTION 6.3.

We can eliminate ρ form the two relations

$$\cosh(v) = \rho,\ \mathfrak{w}_1 = 2e^{-v} = 4\rho b_0,$$

and we get

$$2b_0 \cosh(v) = e^{-v} \text{ or } b_0 = \frac{1}{1+e^{2v}}.$$

Hence

$$e^v = \sqrt{\frac{1}{b_0} - 1} = \sqrt{\frac{1-b_0}{b_0}},$$

and

$$\rho(b_0) = \frac{1}{2}\left(\sqrt{\frac{1-b_0}{b_0}} + \sqrt{\frac{b_0}{1-b_0}}\right)$$

From this follows (after some computations)

$$m = -b_0\partial_{b_0}\Phi = -b_0\rho^{-1}\frac{d\rho}{db_0} = \frac{2b_0-1}{2b_0-2}.$$

We also have

$$\sum_{p=1}^{\infty} w_p^2 = 4\sum_{p=1}^{\infty} e^{-2pv} = 4\frac{e^{-2v}}{1-e^{-2v}} = \frac{4}{e^{2v}-1} = \frac{4}{1/b_0 - 2} = \frac{4b_0}{1-2b_0},$$

hence

$$1 + \frac{1}{4b_0b_1}\sum_{p=1}^{\infty} w_p^2 = 1 + \frac{1}{1-2b_0} = \frac{2-2b_0}{1-2b_0}$$

and, as expected $m = (1-2b_0)\,/\,(2-2b_0)$. □

• A.12 COMPUTATIONS OF SUBSECTION 6.3.1.

We have

$$b_p b_{p+1} = \frac{(p+2a)(p+2a+1)}{(p+2a-1)(p+2a+2)}.$$

Using formulas **15.1.13, 15.1.14**, and **15.2.12** in [1] one gets

$$F(a+p/2-1/2, a+p/2; 2a+p+1; z) =$$

$$2^{2a+p}(1+\sqrt{1-z})^{-2a-p+1}\frac{(a+p/2-1/2)\sqrt{1-z} - a - p/2 + 1/2 + z(a+p/2)}{(a+p/2+1/2)z}.$$

Thus, from Theorem 11

$$\mathfrak{w}_p(\rho) = 2\rho^{-p}(1+\sqrt{1-\rho^{-2}})^{-p}\frac{(a+p/2-1/2)\sqrt{1-\rho^{-2}} - a - p/2 + 1/2 + \rho^{-2}(a+p/2)}{(a+p/2+1/2)}\times$$

$$\frac{a+1/2}{(a-1/2)\sqrt{1-\rho^{-2}} - a + 1/2 + a\rho^{-2}}\sqrt{\frac{a(2a+p+1)}{(2a+p-1)(a+1)}}$$

$$=: As^p\frac{\alpha p + \beta}{\sqrt{(2a+p-1)(2a+p+1)}}$$

where

$$A := 4\frac{a+1/2}{(a-1/2)\sqrt{1-\rho^{-2}}-a+1/2+a\rho^{-2}}\sqrt{\frac{a}{a+1}}$$

$$\alpha := \frac{1}{2}\left(\sqrt{1-\rho^{-2}}-1+\rho^{-2}\right)\ ;\ \beta := \left(a-\frac{1}{2}\right)\sqrt{1-\rho^{-2}}-a+1/2+a\rho^{-2}$$

$$\text{and } s =: \frac{1}{\rho(1+\sqrt{1-\rho^{-2}})}.$$

Therefore $b_0^c = \mathfrak{w}_1(1)/\left(4b_1\right) = a/\left(2a+1\right)$, and, with C_1, C_2 some explicit constants

$$\begin{aligned}\sum_{p=1}^{\infty}\mathfrak{w}_p\left(\rho\right)^2 &= A^2\sum_{p=1}^{\infty}s^{2p}\frac{(\alpha p+\beta)^2}{(2a+p+1)(2a+p-1)} \\ &= \frac{A^2}{2}\sum_{p=1}^{\infty}s^{2p}\left(\frac{\alpha^2 p+C_1}{2a+p+1}+\frac{\alpha^2 p+C_2}{2a+p-1}\right).\end{aligned}$$

Putting $x=s^2$, $v=2a\pm 1$ and using

$$\begin{aligned}\sum_{p=0}^{\infty}\frac{x^p}{v+p} &= \sum_{p=0}^{\infty}x^p\frac{\Gamma(v+p)\Gamma(p+1)}{\Gamma(v+p+1)p!} = \frac{\Gamma(v)}{\Gamma(v+1)}\sum_{p=0}^{\infty}x^p\frac{(1)_p(v)_p}{(v+1)_p p!} \\ &= \frac{\Gamma(v)}{\Gamma(v+1)}F(1,v;v+1;x),\ \text{and} \\ \sum_{p=1}^{\infty}\frac{px^p}{v+p} &= \frac{\Gamma(v+1)}{\Gamma(v+2)}xF(2,v+1;v+2;x),\end{aligned}$$

we get $\sum_{p=1}^{\infty}\mathfrak{w}_p\left(\rho\right)^2$ in terms of Gauss hypergeometric functions, consistently with (4.1) and

$$m(\rho,b_0) = \frac{1}{2+\frac{a(a+1/2)F(a+1,a+3/2;2a+3;\rho^{-2})}{(2a+2)2\rho^4 b_0 b_1 F(a-1/2,a;2a+1;\rho^{-2})}-\frac{2a(a-1/2)F(a+1/2,a+1;2a+2;\rho^{-2})}{\rho^2(2a+1)F(a-1/2,a;2a+1;\rho^{-2})}}.$$

• $A.13$ VERIFICATION OF (4.1) FOR $b_0 = b_0^c$ IN THE HYPERGEOMETRIC MODEL.

- Let $1/2 < s < 3/2$.

In that case, it follows from the asymptotic $\mathfrak{w}_p(1) = \mathcal{O}(1)\cdot p^{1-s}$ that $m=0$ as expected.

- Let $s > 3/2$. We have

$$\begin{aligned}&\sum_{p=1}^{\infty}(2a+p+s-2)\frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)} \\ &= \sum_{p=1}^{\infty}(2a+p-1)\frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)}+(s-1)\sum_{p=1}^{\infty}\frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)} \\ &= \sum_{p=1}^{\infty}\frac{\Gamma(2a+p)}{\Gamma(2a+2s+p-2)}+(s-1)\sum_{p=1}^{\infty}\frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)}.\end{aligned}$$

On one hand

$$\sum_{p=1}^{\infty} \frac{\Gamma(2a+p)}{\Gamma(2a+2s+p-2)} = \sum_{p=1}^{\infty} \frac{\Gamma(2a+p)\,\Gamma(p+1)}{\Gamma(2a+2s+p-2)} \frac{1}{p!}$$

$$= \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \sum_{p=1}^{\infty} \frac{(2a)_p\,(1)_p}{(2a+2s-2)_p} \frac{1}{p!} = \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \left(F(2a,1;2a+2s-2;1)-1\right)$$

$$= \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \left(\frac{\Gamma(2a+2s-2)\,\Gamma(2s-3)}{\Gamma(2s-2)\,\Gamma(2a+2s-3)} - 1 \right)$$

$$= \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \left(\frac{2a+2s-3}{2s-3} - 1 \right) = \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \frac{2a}{2s-3}.$$

Similarly

$$\sum_{p=1}^{\infty} \frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)} = \sum_{p=1}^{\infty} \frac{\Gamma(2a+p-1)\Gamma(p+1)}{\Gamma(2a+2s+p-2)} \frac{1}{p!}$$

$$= \frac{\Gamma(2a-1)}{\Gamma(2a+2s-2)} \sum_{p=1}^{\infty} \frac{(2a-1)_p\,(1)_p}{(2a+2s-2)_p} \frac{1}{p!}$$

$$= \frac{\Gamma(2a-1)}{\Gamma(2a+2s-2)} \left(F(2a-1,1;2a+2s-2;1)-1\right)$$

$$= \frac{\Gamma(2a-1)}{\Gamma(2a+2s-2)} \left(\frac{\Gamma(2a+2s-2)\,\Gamma(2s-2)}{\Gamma(2s-1)\,\Gamma(2a+2s-3)} - 1 \right)$$

$$= \frac{\Gamma(2a-1)}{\Gamma(2a+2s-2)} \left(\frac{2a+2s-3}{2s-2} - 1 \right) = \frac{\Gamma(2a-1)}{\Gamma(2a+2s-2)} \frac{2a-1}{2s-2}$$

$$= \frac{\Gamma(2a)}{\Gamma(2a+2s-2)(2s-2)}.$$

Hence,

$$\sum_{p=1}^{\infty} (2a+p+s-2) \frac{\Gamma(2a+p-1)}{\Gamma(2a+2s+p-2)}$$

$$= \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \frac{2a}{2s-3} + (s-1) \frac{\Gamma(2a)}{\Gamma(2a+2s-2)(2s-2)}$$

$$= \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \left(\frac{2a}{2s-3} + \frac{1}{2} \right) = \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \frac{4a+2s-3}{2\,(2s-3)}.$$

Therefore, from the expression of $\mathfrak{w}_p(1)$ given at the end of the Proof of Theorem 11

$$\sum_{p=1}^{\infty} \mathfrak{w}_p(1)^2 = \frac{2a+s-1}{(a+s-1)^2} \frac{\Gamma(2a+2s-1)}{\Gamma(2a)} \frac{\Gamma(2a)}{\Gamma(2a+2s-2)} \frac{4a+2s-3}{2(2s-3)}$$

$$= \frac{2a+s-1}{(a+s-1)^2} (2a+2s-2) \frac{4a+2s-3}{2(2s-3)}$$

On the other hand

$$\mathfrak{w}_1(1) = \frac{s+2a-1}{s+a-1}$$

hence

$$4b_0^c b_1 = \mathfrak{w}_1(1) = \frac{s+2a-1}{s+a-1} \text{ and}$$

$$1+\frac{1}{4b_0^c b_1}\sum_{p=1}^{\infty}\mathfrak{w}_p(1)^2 = 1+\frac{4a+2s-3}{2s-3} = 2+\frac{2a}{s-3/2}.$$

We finally get

$$m = 1/\left(2+\frac{2a}{(s-3/2)}\right),$$

which is precisely the expression of Proposition12.(iii), derived in Appendix $A.10$ using a direct application of the formula. □

• $A.14$ **PROOF of the UNIVERSALITY of CRITICAL INDICES.**
- Sketch of the Proof of Theorem 9.

We want to find the minimal positive solution of

$$Rw = (1+\epsilon)w - \mathbf{1}_{n=1} \tag{8.13}$$

where

$$(Rw)_n = \frac{w_{n+1}}{2\sqrt{b_n b_{n+1}}} + \frac{w_{n-1}}{2\sqrt{b_n b_{n-1}}}\mathbf{1}_{n>1}.$$

For $\epsilon > 0$, we define the integer $N = N(\epsilon)$ by

$$N(\epsilon) = \left[\frac{1}{\sqrt{\epsilon}}\right].$$

The proof of Theorem 9 uses three zones.

Zone 1. We first consider $n \ll N(\epsilon)$. In that case we start by neglecting ϵ in equation (8.13). Note that indeed in that regime $|1-b_n| \gg \epsilon$. We prove that w_n is well approximated by $w_n^0 = \mathfrak{w}_n$. Recall (see Proposition 15) that $\mathfrak{w}_n(1)$ behaves like $C_1 n^\alpha$ for large n, with C_1 a constant independent of n and $\alpha = \alpha_-$ the solution smaller than 1/2 of (5.3). Recall that for $-3/8 < \mathrm{w} < 1/8$ we have $-1/2 < \alpha_- < 1/2$.

Zone 3. For $n \gg N(\epsilon)$, we will use a refined version of Lemma 16 to prove that w_n is well approximated by

$$C_3\epsilon^{-\alpha/2}e^{-nk},$$

where C_3 is some constant independent of n and ϵ to be fixed later on, and $k = \cosh^{-1}(1+\epsilon) \sim \sqrt{2\epsilon}$.

Zone 2. For $n \approx N(\epsilon)$, we introduce the scaled variable $x = n\sqrt{\epsilon}$. We then look in this range for a solution w_n of the form $f(n\sqrt{\epsilon})$. Using this ansatz in equation (8.13) and expanding, we get (at dominant order) for f the equation

$$f''(x) + \left(\frac{2\mathrm{w}}{x^2} - 2\right)f(x) = 0.$$

See below for the details in zone 2. According to formula **8.491.5** in [9], (taken at a purely imaginary argument) we get

$$f(x) = A\sqrt{x}K_\nu(\sqrt{2}x) + B\sqrt{x}I_\nu(\sqrt{2}x)$$

for some constants A, B, and ν given by

$$\nu = \frac{\sqrt{1-8\mathrm{w}}}{2} = \frac{1}{2} - \alpha.$$

Note that since $\alpha = \alpha_-$ and $-1/2 < \alpha_- < 1/2$, we have $0 < \nu < 1$. We look for a solution which tends to zero at infinity exponentially fast. In the sequel we will therefore take $B = 0$, and $A = \epsilon^{-\alpha/2}$ for the homogeneity of the matching, leading to

$$f(x) = \epsilon^{-\alpha/2}\sqrt{x}K_\nu(\sqrt{2}x). \tag{8.14}$$

Recall that (see for example [1].**9.7.2** and [1].**9.6.9**) $K_\nu(x) = K_{-\nu}(x)$ and for $\nu > 0$

$$K_\nu(x) = \begin{cases} 2^{\nu-1}\,\Gamma(\nu)\,x^{-\nu}\,(1+o(x)) & \text{for } 0 < x \le 1\,, \\ \sqrt{\pi/2}\,x^{-1/2}\,e^{-x}(1+o(1/x)) & \text{for } x \ge 1\,. \end{cases}$$

Therefore,

$$f(x) = \begin{cases} \mathcal{O}(1)\,\epsilon^{-\alpha/2}\,x^\alpha & \text{for } 0 < x \le 1\,, \\ \mathcal{O}(1)\,\epsilon^{-\alpha/2}\,e^{-x\sqrt{2}} & \text{for } x \ge 1\,. \end{cases}$$

or equivalently

$$f\left(n\sqrt{\epsilon}\right) = \begin{cases} \mathcal{O}(1)\,n^\alpha & \text{for } 1 \le n \le N\,, \\ \mathcal{O}(1)\,\epsilon^{-\alpha/2}\,e^{-n\sqrt{2\epsilon}} & \text{for } n \ge N\,. \end{cases}$$

These two asymptotic estimates allow the matching at the dominant order. The complete proof of the matching is done using estimates on the remainders.

- Matching equations.

We define V as the unique solution of the equation

$$RV = (1+\epsilon)V$$

satisfying $V_1 = 1$. This solution is unique since the recursion is of order two except for $n = 1$. We define $(G)_{n\ge 2}$ as the unique solution of the equation

$$RG = (1+\epsilon)G,$$

satisfying (recalling $N = N(\epsilon)$)

$$G_N = N^{1-\alpha},\; G_{N+1} = (N+1)^{1-\alpha}.$$

This solution is unique since the recursion is of order two.

The solutions W, F and H of

$$RX = (1+\epsilon)X - \mathbf{1}_{n=1}$$

and the numbers $1 < M < N_1 < N < N_2 < M' < \infty$ are given in Propositions 36, 37, 33; see (8.23), (8.19) and (8.17). Within each zone, we will construct solutions W_n^i, $i = 1, ..., 3$ as follows (for some constants A, B, C, D):

- Zone 1 : $n \le N_1$, solution $W_n^1 = W_n + AV_n$.
- Zone 2 : $M \le n \le M'$, solution $W_n^2 = BF_n + CG_n$.
- Zone 3 : $n \ge N_2$, solution $W_n^3 = DH_n$.
- The matching points are L and $L+1$, L' and $L'+1$, with $M < L < N_1 < N < N_2 < L' < M'$ where

$$M = O(1)\,,\; L = \left[\frac{\epsilon^{-1/2}}{\log(\epsilon^{-1})}\right],\; N_1 = \left[\widetilde{C}\epsilon^{-1/2}\right],\; N = \left[\epsilon^{-1/2}\right],\; N_2 = \left[1 + 3C'\widetilde{C}\epsilon^{-1/2}\right],$$

$$L' = \left[N(1 + \frac{\gamma\log\left(\epsilon^{-1}\right)}{2\sqrt{2}}\right] \text{ and } M' = \left[N + \frac{\zeta\epsilon^{-1/2}}{2\sqrt{2}}\log\left(\epsilon^{-1}\right)\right].$$

This is summarized in figure 3.

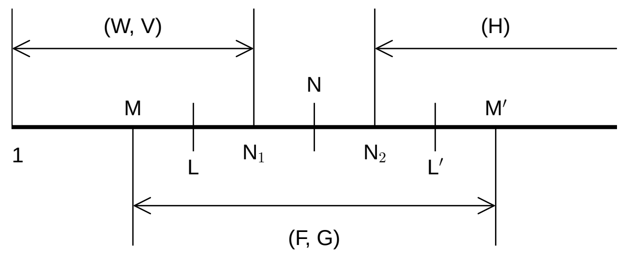

FIGURE 3. Domains of solutions and matching points.

The matching conditions therefore are

$$W_L + AV_L = BF_L + CG_L,\ W_{L+1} + AV_{L+1} = BF_{L+1} + CG_{L+1},$$

$$BF_{L'} + CG_{L'} = DH_{L'},\ BF_{L'+1} + CG_{L'+1} = DH_{L'+1}.$$

These matching conditions can be written in matrix form, namely

$$\begin{pmatrix} -V_L & F_L & G_L & 0 \\ -V_{L+1} & F_{L+1} & G_{L+1} & 0 \\ 0 & F_{L'} & G_{L'} & -H_{L'} \\ 0 & F_{L'+1} & G_{L'+1} & -H_{L'+1} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} W_L \\ W_{L+1} \\ 0 \\ 0 \end{pmatrix}.$$

Define the discrete Wronskian sequence (after Josef Hoëné-Wronski and Casorati) $W\left(X,Y\right)_n$ by

$$W\left(X,Y\right)_n = X_{n+1}Y_n - X_nY_{n+1}.$$

After some computations using Cramer's rule, one gets

$$A = \frac{W\left(G,H\right)_{L'} W\left(F,W\right)_L - W\left(F,H\right)_{L'} W\left(G,W\right)_L}{W\left(G,H\right)_{L'} W\left(V,F\right)_L + W\left(F,H\right)_{L'} W\left(G,V\right)_L} \tag{8.15}$$

$$B = \frac{W\left(G,H\right)_{L'} W\left(V,W\right)_L}{W\left(G,H\right)_{L'} W\left(V,F\right)_L + W\left(F,H\right)_{L'} W\left(G,V\right)_L},$$

$$\frac{C}{B} = -\frac{W\left(F,H\right)_{L'}}{W\left(G,H\right)_{L'}} \text{ and } D = -\frac{W\left(F,G\right)_{L'} W\left(V,W\right)_L}{W\left(G,H\right)_{L'} W\left(V,F\right)_L + W\left(F,H\right)_{L'} W\left(G,V\right)_L}.$$

We have:

Theorem 31. *There exist constants $c > 1$ and $\epsilon_0 > 0$, such that for any $\epsilon \in]0, \epsilon_0]$, for any $L = \left[\frac{\epsilon^{-1/2}}{\log(\epsilon^{-1})}\right]$ and for $L' = \left[N(1 + \frac{\gamma \log(\epsilon^{-1})}{2\sqrt{2}})\right]$, with*

$$\gamma = \inf\left\{\zeta/2, 2\sqrt{2}\bar{C}'\right\},$$

(see Proposition 37 for the definition of $\bar{C}'$), we have

$$\frac{1}{c} \leq W(G,H)_{L'}\,,\ W(V,F)_L\,,\ W(V,W)_L\,,\ W(G,F)_{L'} \leq c$$

and $|W(G,W)_L| \leq c$. Moreover

$$|W(G,V)_L| \leq \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2},$$
$$|W(F,W)_L| \leq \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha},$$
$$|W(F,H)_{L'}| \leq \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha+\gamma}.$$

Proof:

Estimate of $W(V,W)_L$. Note that (since $V_1 = 1$)

$$V_2 = 2\sqrt{b_1 b_2}(1+\epsilon)$$

and

$$W_2 = 2\sqrt{b_1 b_2}(1+\epsilon)W_1 - 2\sqrt{b_1\, b_2}.$$

Therefore

$$W(V,W)_1 = V_2 W_1 - W_2 V_1 = 2\sqrt{b_1\, b_2}.$$

The bound on $W(V,W)_L$ follows from Lemma 14 and

$$0 < \prod_{j=1}^{\infty} b_j < \infty.$$

Estimate of $W(G,F)_{L'}$. Using Lemma 14, we get

$$W(G,F)_{L'} \approx W(G,F)_N\,.$$

We use equation (8.28) with $x_n = n^{1-\alpha}$, $\delta_n^x = 0$ (since $n = N$ or $N+1$), $y_n = f(n\sqrt{\epsilon})$. We obtain using Proposition 37

$$W(G,F)_N = W(G,y)_N(1+\delta_N^y) + \mathcal{O}(1)\cdot N\epsilon^{1/2+\zeta/2}.$$

$$\begin{aligned}
W(G,y)_N &= (N+1)^{1-\alpha}f(N\sqrt{\epsilon}) - N^{1-\alpha}f((N+1)\sqrt{\epsilon}) \\
&= N^{1-\alpha}\left(\left(1+\frac{1}{N}\right)^{1-\alpha} f(N\sqrt{\epsilon}) - f((N+1)\sqrt{\epsilon})\right) \\
&= N^{1-\alpha}\left(\frac{1-\alpha}{N}f(N\sqrt{\epsilon}) + f(N\sqrt{\epsilon}) - f((N+1)\sqrt{\epsilon}) + \mathcal{O}(1)\cdot N^{-2}f(N\sqrt{\epsilon})\right) \\
&= N^{1-\alpha}\left(\frac{1-\alpha}{N}f(N\sqrt{\epsilon}) - \sqrt{2\epsilon}f'(N\sqrt{\epsilon}) - \frac{\epsilon}{2}f''(\xi) + \mathcal{O}(1)\cdot N^{-2}f(N\sqrt{\epsilon})\right)
\end{aligned}$$

for some $\xi \in [N\sqrt{\epsilon}, (N+1)\sqrt{\epsilon}]$. This implies

$$W(G,y)_N = N^{-\alpha}\left((1-\alpha)f(N\sqrt{\epsilon}) - \sqrt{\epsilon}Nf'(N\sqrt{\epsilon})\right) + \mathcal{O}(1)\cdot\epsilon^{1/2}.$$

Using formula **8.472.2** in [9] and $\alpha = 1/2 - \nu$, we have

$$(1-\alpha)f(x) - xf'(x) = \epsilon^{-\alpha/2}x^{3/2}K_{\nu+1}(x).$$

Therefore

$$W(G,y)_N = N^{-\alpha}\epsilon^{-\alpha/2}(N\sqrt{2\epsilon})^{3/2}K_{\nu+1}(N\sqrt{2\epsilon}) = 2^{3/4}K_{\nu+1}(\sqrt{2}) + \mathcal{O}(1)\cdot\epsilon^{1/2}.$$

We obtain

$$W(G,F)_N = 2^{3/4}K_{\nu+1}(\sqrt{2}) + o(1) \tag{8.16}$$

and the bound on $W(G,F)_{L'}$ follows.

Estimate of $W(F,H)_{L'}$.

We use equation (8.28) with $x_n = \epsilon^{1/4-\alpha/2}\sqrt{n}K_\nu(n\sqrt{2\epsilon}) = f(n\sqrt{\epsilon})$ and $y_n = \epsilon^{-\alpha/2}e^{-kn}$. We get using Propositions 37 and 33

$$W(F,H)_{L'} = W(x,y)_{L'}\left(1+\delta_{L'}^2+\delta_{L'}^3+\delta_{L'}^2\delta_{L'}^3\right) + R_{L'}^{F,H}$$

with

$$\begin{aligned}|R_{L'}^{F,H}| &\le \mathcal{O}(1)\cdot\epsilon^{-\alpha}e^{-(L'-N)(k+\sqrt{2\epsilon})}\left(\epsilon^{-1/2}L'^{-2} + \epsilon^{1/2+\zeta/2}e^{2(L'-N)\sqrt{2\epsilon}}\right)\\ &\le \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha}\left(e^{-2(L'-N)\sqrt{2\epsilon}} + \epsilon^{\zeta/2}\right)\end{aligned}$$

for $L' \le \mathcal{O}(1)\cdot N\log\epsilon^{-1}$.

Since $|\delta_x| \le 1/2$ and $|\delta_y| \le 1/2$, we have

$$-\frac{3}{4} \le \delta_x + \delta_y + \delta_x\delta_y \le \frac{5}{4}.$$

Finally

$$W(x,y)_{L'} = \epsilon^{-\alpha/2}e^{-k(L'-N)}\left(f((L'+1)\sqrt{\epsilon}) - e^{-k}f(L'\sqrt{\epsilon})\right).$$

$$\begin{aligned}\left|f((L'+1)\sqrt{\epsilon}) - e^{-k}f(L'\sqrt{\epsilon})\right| &\le \left(1-e^{-k}\right)f(L'\sqrt{\epsilon}) + \mathcal{O}(1)\cdot\sqrt{\epsilon}\,|f'(\xi)|\\ &\le \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha}e^{-2(L'-N)\sqrt{2\epsilon}}.\end{aligned}$$

We get

$$|W(F,H)_{L'}| \le \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha}\left(e^{-2(L'-N)\sqrt{2\epsilon}} + \epsilon^{\zeta/2} + e^{-2(L'-N)\sqrt{2\epsilon}}\right).$$

With our choice of L', for ϵ small enough we get

$$|W(F,H)_{L'}| \le \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha+\gamma}.$$

Estimate of $W(G,H)_{L'}$.

We use equation (8.31) with $x_n = F_n$, $y_n = G_n$, and $z_n = H_n$ and $p = N$. We get

$$W(H,G)_{L'} = W(H,F)_{L'}\left(\frac{G_p}{F_p} + \frac{W(G,F)_p}{\sqrt{b_{p+1}b_p}}\tilde{G}_{L'+1}\right) - H_{L'}\frac{W(G,F)_p}{\sqrt{b_{p+1}\,b_p}}\frac{\sqrt{b_{L'+1}\,b_{L'}}}{F_{L'}},$$

$$\tilde{G}_{L'+1} = \sum_{l=N}^{L'}\frac{\sqrt{b_{l+1}b_l}}{F_lF_{l+1}} \le \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2},$$

$$\frac{G_N}{F_N} = \mathcal{O}(1)\cdot\frac{N^{1-\alpha}}{N^\alpha} = \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2}.$$

Using (8.16), Propositions 33 and 37 and the bound on $W_{L'}$ and $H_{L'}/F_{L'} = 1+o(1)$, we get

$$W(H,G)_{L'} = -\sqrt{\frac{2}{\pi}}2^{3/4}K_{\nu+1}(\sqrt{2}) + o(1).$$

Estimate of $W(F,W)_L$**.** We use equation (8.28) with $x_n = f(n\sqrt{\epsilon})$ and $y_n = n^\alpha$. We have

$$W(x,y)_L = f((L+1)\sqrt{\epsilon})L^\alpha - (L+1)^\alpha f(L\sqrt{\epsilon}) = L^\alpha\left(\sqrt{\epsilon}f'(L\sqrt{\epsilon}) - \frac{\alpha}{L}f(L\sqrt{\epsilon})\right)$$
$$+L^\alpha\left(\frac{\mathcal{O}(1)}{L^2}f(L\sqrt{\epsilon}) + \epsilon f''(\xi)\right)$$

for some $\xi \in [L\sqrt{\epsilon},(L+1)\sqrt{\epsilon}]$.

$$\begin{aligned}
L^\alpha\left(\sqrt{\epsilon}f'(L\sqrt{\epsilon}) - \frac{\alpha}{L}f(L\sqrt{\epsilon})\right) &= L^{\alpha-1}\left(L\sqrt{\epsilon}f'(L\sqrt{\epsilon}) - \left(\frac{1}{2}-\nu\right)f(L\sqrt{\epsilon})\right)\\
&\underset{x=L\sqrt{\epsilon}}{=} L^{\alpha-1}\left(xf'(x) - \left(\frac{1}{2}-\nu\right)f(x)\right)\\
= L^{\alpha-1}\epsilon^{-\alpha/2}\left(\frac{\sqrt{x}}{2}K_\nu(\sqrt{2}x) + \sqrt{2}x^{3/2}K'_\nu(\sqrt{2}x) - \left(\frac{1}{2}-\nu\right)\sqrt{x}K_\nu(\sqrt{2}x)\right)&\\
= L^{\alpha-1}\epsilon^{-\alpha/2}\sqrt{x}\left(\sqrt{2}xK'_\nu(\sqrt{2}x) + \nu K_\nu(\sqrt{2}x)\right) = \sqrt{2}L^{\alpha-1}\epsilon^{-\alpha/2}x^{3/2}K_{\nu-1}(\sqrt{2}x)&\\
= \mathcal{O}(1)\cdot L^{\alpha+1/2}\epsilon^{3/4-\alpha/2}K_{1-\nu}(L\sqrt{2\epsilon}) = \mathcal{O}(1)\cdot L^{\alpha+1/2}\epsilon^{3/4-\alpha/2}L^{\nu-1}\epsilon^{\nu/2-1/2}&\\
= \mathcal{O}(1)\cdot L^{\alpha+1/2}\epsilon^{3/4-\alpha/2}L^{-\alpha-1/2}\epsilon^{-1/4-\alpha/2} = \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha}&
\end{aligned}$$

by **8.472.1** in [9]**.** We have

$$\left|L^\alpha\left(\frac{\mathcal{O}(1)}{L^2}f(L\sqrt{\epsilon}) + \epsilon f''(\xi)\right)\right| \le \mathcal{O}(1)\cdot L^{2\alpha-2}.$$

Using 36 and 37 we get for the second term in (8.28)

$$\begin{aligned}
&y_L x_{L+1}\left((\delta^x_{L+1} - \delta^x_L)(1+\delta^y_L) - (\delta^y_{L+1} - \delta^y_L)(1+\delta^x_L)\right)\\
&\quad = \mathcal{O}(1)\cdot L^{2\alpha}\left(\epsilon L + L^{-1-\zeta}\right) \le \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha}.
\end{aligned}$$

Collecting all the terms we get

$$|W(F,W)_L| \le \mathcal{O}(1)\cdot\epsilon^{1/2-\alpha}.$$

Estimate of $W(G,W)_L$**.** We will use (8.31) with $z = W$, $y = G$, $x = F$, $p = N$ and $n = L$. We have

$$\frac{y_p}{x_p} = \frac{G_N}{F_N} = \mathcal{O}(1)\cdot N^{1-2\alpha} = \mathcal{O}(1)\cdot\epsilon^{-1/2+\alpha}.$$

Next $W(y,x)_p = W(G,F)_N = \mathcal{O}(1)$ as seen from the estimation of $W(G,F)_L$ and

$$\widetilde{y}_{n+1} = -\sum_{l=L}^{N}\frac{\sqrt{b_{l+1}b_l}}{F_lF_{l+1}} = \mathcal{O}(1)\cdot N^{1-2\alpha} = \mathcal{O}(1)\cdot\epsilon^{-1/2+\alpha}.$$

We also have

$$\frac{z_n}{x_n} = \frac{W_L}{F_L} = \mathcal{O}(1),$$

using Proposition 36 and Lemma 14. The estimate $|W(G,W)_L| = \mathcal{O}(1)$ follows using the above estimates and the estimate of $W(W,F)_L$.

Estimate of $W(V,F)_L$.
We use equation (8.31) with $x_n = W_n$, $y_n = V_n$, $z_n = F_n$ and $p = 1$. We get

$$W(F,V)_L = W(F,W)_L\left(\frac{V_1}{W_1} + \frac{W(V,W)_1}{\sqrt{b_2 b_1}}\tilde{V}_{L+1}\right) - F_L\frac{W(V,W)_1}{\sqrt{b_2 b_1}}\frac{\sqrt{b_{L+1}b_L}}{W_L}.$$

$$\tilde{V}_{L+1} = \sum_{l=1}^{L}\frac{\sqrt{b_{l+1}b_l}}{W_l W_{l+1}} = \mathcal{O}(1)\cdot L^{1-2\alpha},\ W(V,W)_1 = 2\sqrt{b_1 b_2}.$$

Using the above estimate of $W(F,W)_L$, Propositions 36 and 37, we get

$$W(V,F)_L = 2\frac{\epsilon^{-\alpha/2}\sqrt{L\sqrt{\epsilon}}K_\nu(L\sqrt{2\epsilon})}{\tilde{C}L^\alpha} + \mathcal{O}(1)\cdot\left(\epsilon^{1/2-\alpha}L^{1-2\alpha} + L^{-2} + \epsilon^{1/2-\alpha}\right)$$

$$= \frac{2^{\nu/2}\Gamma(\nu)}{\tilde{C}}\left(1 + o(1)\right).$$

Estimate of $W(G,V)_L$.
We first use equation (8.31) with $x_n = F_n$, $y_n = G$, $z_n = V_n$ and $p = N$. We obtain

$$W(V,G)_L = W(V,F)_L\left(\frac{G_N}{F_N} + \frac{W(G,F)_N}{\sqrt{b_{N+1}b_N}}\tilde{G}_{L+1}\right) - V_L\frac{W(G,F)_N}{\sqrt{b_{N+1}b_N}}\frac{\sqrt{b_{L+1}b_L}}{F_L}.$$

We have the estimate

$$\frac{G_N}{F_N} - \frac{W(G,F)_N}{\sqrt{b_{N+1}b_N}}\sum_{l=L+1}^{N-1}\frac{\sqrt{b_{l+1}b_l}}{F_l F_{l+1}} = \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2}.$$

We also have

$$\frac{V_L}{F_L} = \frac{W_L}{F_L}\frac{V_L}{W_L}$$

and use (8.29) with $y_n = V_n$, $x_n = W_n$ and $p = 1$. We obtain

$$\frac{V_L}{W_L} = \frac{V_1}{W_1} + \frac{W(V,W)_1}{\sqrt{b_2 b_1}}\sum_{l=1}^{L-1}\frac{\sqrt{b_{l+1}b_l}}{W_l W_{l+1}} = \mathcal{O}(1)\cdot L^{1-2\alpha}.$$

Combining the above estimates we get

$$|W(G,V)_L| \le \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2}.\ \square$$

Corollary 32. *Under the hypotheses and notation of Theorem 31, we have*

$$\mathfrak{w}_n(1+\epsilon) = \begin{cases} W_n + A\,V_n & \text{for } 1 \le n \le L+1 \\ B\,F_n + C\,G_n & \text{for } L \le n \le L'+1 \\ D\,H_n & \text{for } L' \le n\ , \end{cases}$$

with A, B, C, D given by 8.15.

Proof: We define a sequence (w_n) by

$$w_n = \begin{cases} W_n + A\,V_n & \text{for } 1 \le n \le L+1 \\ B\,F_n + C\,G_n & \text{for } L \le n \le L'+1 \\ D\,H_n & \text{for } L' \le n\ . \end{cases}$$

Using Propositions 36, 37, 33, and the fact that A, B, C, D solve the matching conditions, we deduce that (w_n) satisfies $Rw = (1+\epsilon)\,w - \mathbf{1}_{n=1}$. From Proposition 33, we also know that this sequence has the right asymptotics at infinity. It remains to prove that $w_n > 0$ for all $n \ge 1$ and the result $w_n = \mathfrak{w}_n(1+\epsilon)$ will follow from (3.4).

Using Theorem 31 we get

$$\left|W\left(F,H\right)_{L'}\,W\left(G,V\right)_{L}\right| \le \mathcal{O}(1)\cdot\epsilon^{\gamma}.$$

therefore $D > 0$ and for $\epsilon > 0$ small enough

$$B \ge c^{-2} + \mathcal{O}(1)\cdot\epsilon^{\gamma} > 0.$$

From $D > 0$ we have $w_n > 0$ for $n \ge L'$.

From equation (8.29) we have (with $p = 1$)

$$\left|\frac{V_n}{W_n}\right| \le \mathcal{O}(1)\cdot n^{1-2\alpha}.$$

Using Theorem 31, this implies that for $L < \mathcal{O}(1)\cdot\epsilon^{-1/2}$ and $\epsilon > 0$ small enough

$$\sup_{1\le n\le L}\left|A\frac{V_n}{W_n}\right| \le \frac{1}{2}.$$

From the positivity of W_n, this implies $w_n > 0$ for $1 \le n \le L$.

For $L \le n \le N$, we use equation (8.30) (with $p = N$) giving

$$\left|\frac{G_n}{F_n}\right| \le \mathcal{O}(1)\cdot N^{1-2\alpha} \le \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2}.$$

For $N \le n \le L'$, we use equation (8.29) with $p = N$ also giving

$$\left|\frac{G_n}{F_n}\right| \le \mathcal{O}(1)\cdot\epsilon^{\alpha-1/2}.$$

This implies using Theorem 31

$$\sup_{L\le n\le L'}\left|\frac{C}{B}\frac{G_n}{F_n}\right| \le \mathcal{O}(1)\cdot\epsilon^{\gamma}.$$

Therefore for $\epsilon > 0$ small enough, we conclude that $w_n > 0$ for any $L \le n \le L'$. □

- Proof of the main result (equation (5.2)**).**

We can split the sum

$$\sum_{p=1}^{\infty}\mathfrak{w}_p(1+\epsilon)^2 = \sum_{p=1}^{L}\mathfrak{w}_p(1+\epsilon)^2 + \sum_{p=L+1}^{L'}\mathfrak{w}_p(1+\epsilon)^2 + \sum_{p=L'}^{\infty}\mathfrak{w}_p(1+\epsilon)^2.$$

The result follows from Corollary 32 using Propositions 36, 37 and 33. Note that each of the three terms of the sum contribute an order $\epsilon^{-\theta}$. □

We now come to the construction of the solutions in the 3 different zones that were just used in the proof of equation (5.2) from the matching conditions.

- Zone 3, $n \gg N(\epsilon)$**.**

Proposition 33. *There exists a constant* $C' > 0$*, such that for any* $\epsilon \in]0,1]$*, and for any*

$$n \geq N_2 = 1 + \left[\frac{3C'}{\sqrt{\epsilon}}\right], \tag{8.17}$$

the equation $Rw = (1+\epsilon)w$*, a unique (positive) solution* (H_n) *such that*

$$H_n = \epsilon^{-\alpha/2} e^{-kn}(1+\delta_n^3),$$

with $k = \cosh^{-1}(1+\epsilon)$*, and* $\lim_{n\to\infty} \delta_n^3 = 0$*. Moreover*

$$\sup_{n \geq N_2} |\delta_n^3| \leq \frac{1}{2},$$

and for any $n \geq N_2$

$$|\delta_n^3| \leq \frac{\mathcal{O}(1)}{n\sqrt{\epsilon}},$$

and

$$|\delta_n^3 - \delta_{n+1}^3| \leq \frac{\mathcal{O}(1)}{n^2\sqrt{\epsilon}}.$$

Proof: We look for a solution of equation (8.13) of the form

$$w_n = \epsilon^{-\alpha/2} e^{-kn}(1+\delta_n).$$

The factor $\epsilon^{-\alpha/2}$ in front is to ensure the homogeneity in the matching. Inserting this ansatz into equation (8.13), we get a recursive equation for δ_n (see [15]). We get

$$\frac{\epsilon^{-\alpha/2} e^{-k(n+1)}(1+\delta_{n+1})}{2\sqrt{b_{n+1}b_n}} + \frac{\epsilon^{-\alpha/2} e^{-k(n-1)}(1+\delta_{n-1})}{2\sqrt{b_{n-1}b_n}} = \epsilon^{-\alpha/2}(1+\epsilon)e^{-kn}(1+\delta_n).$$

This can be rearranged as follows.

$$\frac{e^{-k}\,(1+\delta_{n+1})}{2} + \frac{e^{k}\,(1+\delta_{n-1})}{2} - (1+\epsilon)\,(1+\delta_n) =$$
$$e^{-k}(1+\delta_{n+1})\left(\frac{1}{2} - \frac{1}{2\sqrt{b_{n+1}b_n}}\right) + e^{k}(1+\delta_{n-1})\left(\frac{1}{2} - \frac{1}{2\sqrt{b_{n-1}b_n}}\right).$$

This can also be rewritten (using $1+\epsilon = \cosh(k)$)

$$e^{-k}(\delta_{n+1}-\delta_n) - e^{k}(\delta_n - \delta_{n-1}) =$$
$$2e^{-k}(1+\delta_{n+1})\left(\frac{1}{2} - \frac{1}{2\sqrt{b_{n+1}b_n}}\right) + 2e^{k}(1+\delta_{n-1})\left(\frac{1}{2} - \frac{1}{2\sqrt{b_{n-1}b_n}}\right).$$

or

$$\delta_n - \delta_{n-1} = e^{-2k}(\delta_{n+1}-\delta_n)$$
$$-2e^{-2k}(1+\delta_{n+1})\left(\frac{1}{2} - \frac{1}{2\sqrt{b_{n+1}b_n}}\right) - 2(1+\delta_{n-1})\left(\frac{1}{2} - \frac{1}{2\sqrt{b_{n-1}b_n}}\right).$$

Assuming the sequence (δ_n) converges to zero when n tends to infinity, we get using equation (8.26) with $p = \infty$

$$\delta_n - \delta_{n-1} =$$
$$-\sum_{j=0}^{\infty} e^{-2k(j+1)} \left(e^{-2k}(1+\delta_{n+1+j}) \left(1 - \frac{1}{\sqrt{b_{n+1+j} b_{n+j}}}\right) + (1+\delta_{n-1+j}) \left(1 - \frac{1}{\sqrt{b_{n-1+j} b_{n+j}}}\right)\right).$$

Finally, assuming $\lim_{n\to\infty} \delta_n = 0$ we get using equation (8.27) with $p = \infty$

$$\begin{aligned} \delta_n &= -\sum_{p=n}^{\infty}\sum_{j=0}^{\infty} e^{-2k(j+1)} [e^{-2k}(1+\delta_{p+2+j}) \left(1 - \frac{1}{\sqrt{b_{p+2+j} b_{p+1+j}}}\right) + \\ & \quad (1+\delta_{p+j}) \left(1 - \frac{1}{\sqrt{b_{p+j} b_{p+1+j}}}\right)]. \end{aligned}$$

Observe that the right hand side is the action of an affine operator on δ, denoted by $\mathfrak{T}(\underline{\delta})$. From the asymptotic behavior of (b_n) (see equation (5.1)) we conclude that there is a constant $C > 0$ such that for any $r \geq 1$

$$\left| 1 - \frac{1}{\sqrt{b_{r+1} b_r}} \right| \leq \frac{C}{r^2}.$$

Therefore

$$\left| \sum_{p=n}^{\infty}\sum_{j=0}^{\infty} e^{-2k(j+1)} \left(e^{-2k} \left(1 - \frac{1}{\sqrt{b_{p+2+j} b_{p+1+j}}}\right) + \left(1 - \frac{1}{\sqrt{b_{p+j} b_{p+1+j}}}\right)\right) \right|$$
$$\leq 2C \sum_{p=n}^{\infty}\sum_{j=0}^{\infty} e^{-2kj} \frac{1}{(p+j)^2}.$$

It is easy to show (using $k \approx \sqrt{\epsilon}$) that there exists a constant $C' > 0$ such that for any $\epsilon \in]0,1]$

$$2C \sum_{p=n}^{\infty}\sum_{j=0}^{\infty} e^{-2kj} \frac{1}{(p+j)^2} \leq C' \frac{1}{n\sqrt{\epsilon}}.$$

We now take

$$N_2 > \frac{3C'}{\sqrt{\epsilon}}.$$

Denote by $\mathfrak{B}_3$ the Banach space of bounded sequences on $\{N_2, N_2+1, \ldots\}$ tending to zero at infinity and equipped with the sup norm. It is easy to verify from the above estimate that the affine operator $\mathfrak{T}$ maps $\mathfrak{B}_3$ into itself with

$$\|\mathfrak{T}(\underline{0})\|_{\mathfrak{B}_3} \leq \frac{1}{3} \text{ and } \|D\mathfrak{T}(\underline{0})\|_{\mathfrak{B}_3} \leq \frac{1}{3}.$$

Here $D\mathfrak{T}$ denotes the differential of the map. Therefore, by the contraction mapping principle (see [20]), the equation $\underline{\delta} = \mathfrak{T}(\underline{\delta})$ has a unique fixed point in $\mathfrak{B}_3$ whose norm is at most $1/2$. The last two bounds follow from equations (8.27) and (8.26) using estimates as above. □

- Zone 1, $n \ll N(\epsilon)$.

Denote by $(w^1)_n$ the positive solution of $Rw^1 = w^1 - \mathbf{1}_{n=1}$, assumed to exist from 1-transience of R. In Corollary 23, we showed that $\left(w^1\right)_n$ behaves like n^α for large n but we need here more precise asymptotics.

Proposition 34. *There exist an integer $n_1 > 0$ and a constant $\tilde{C} > 0$ such that for $n > n_1$, w_n^1 satisfies*

$$w_n^1 = \tilde{C} n^\alpha \left(1 + \tilde{\delta}_n^1\right)$$

with

$$\sup_{n>n_1} \left|\tilde{\delta}_n^1\right| < 1/2$$

and

$$\left|\tilde{\delta}_{n+1}^1 - \tilde{\delta}_n^1\right| \leq \mathcal{O}(1) \cdot n^{-1-\zeta}.$$

Proof: We consider the equation $Rw = w$ for large n, and we look for a solution of the form

$$w_n = n^\alpha \left(1 + \delta_n\right).$$

Using this ansatz ($n > 2$), we get

$$\frac{(n+1)^\alpha (1 + \delta_{n+1})}{2\sqrt{b_n b_{n+1}}} + \frac{(n-1)^\alpha (1 + \delta_{n-1})}{2\sqrt{b_n b_{n-1}}} = n^\alpha (1 + \delta_n).$$

This can be rearranged as

$$\frac{(1+1/n)^\alpha}{2\sqrt{b_n b_{n+1}}}(\delta_{n+1} - \delta_n) - \frac{(1-1/n)^\alpha}{2\sqrt{b_n b_{n-1}}}(\delta_n - \delta_{n-1}) = T_n(1 + \delta_n)$$

with

$$T_n = 1 - \frac{(1+1/n)^\alpha}{2\sqrt{b_n b_{n+1}}} - \frac{(1-1/n)^\alpha}{2\sqrt{b_n b_{n+1}}}.$$

We deduce that for $n > 1$ using equation (8.26) with $p = \infty$,

$$h_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}} \frac{(1-1/n)^\alpha}{(1+1/n)^\alpha}, \; g_n = \frac{2\sqrt{b_n b_{n+1}}}{(1+1/n)^\alpha} T_n (1 + \delta_n)$$

we get (assuming $\lim_{n\to\infty} \delta_n = 0$)

$$\delta_{k-1} - \delta_k = 2 \sum_{l=k}^{\infty} \sqrt{\frac{b_{k-1} b_k}{b_{l+1} b_l}} \frac{\sqrt{b_l b_{l+1}}}{(1+1/l)^\alpha} T_l (1 + \delta_l) \prod_{j=k}^{l} \frac{(1+1/j)^\alpha}{(1-1/j)^\alpha}$$

$$\delta_{k-1} - \delta_k = 2 \sum_{l=k}^{\infty} \sqrt{\frac{b_{k-1} b_k}{(1+1/l)^\alpha}} T_l (1 + \delta_l) \prod_{j=k}^{l} \frac{(1+1/j)^\alpha}{(1-1/j)^\alpha}. \tag{8.18}$$

Observe that from $\alpha^2 - \alpha + 2\mathrm{w} = 0$

$$|T_n| \leq \mathcal{O}(1) \cdot n^{-2-\zeta}.$$

Therefore

$$\sum_{l=k}^{\infty} \sqrt{\frac{b_{k-1} b_k}{(1+1/l)^\alpha}} |T_l| \prod_{j=k}^{l} \frac{(1+1/j)^\alpha}{(1-1/j)^\alpha} \leq \mathcal{O}(1) \cdot \sum_{l=k}^{\infty} l^{-2-\zeta} \frac{l^{2\alpha}}{k^{2\alpha}} \leq \mathcal{O}(1) \cdot k^{-1-\zeta}.$$

We choose n_1 such that

$$2\sum_{k=n_1}^{\infty}\sum_{l=k}^{\infty}\sqrt{\frac{b_{k-1}b_k}{(1+1/l)^\alpha}}|T_l|\prod_{j=k}^{l}\frac{(1+1/j)^\alpha}{(1-1/j)^\alpha}<1/3.$$

Define an affine map $\mathfrak{T}\left(\underline{\delta}\right)$ in $\ell^\infty\{n_1,\, n_1+1,\,\ldots\}$ by

$$\mathfrak{T}\left(\underline{\delta}\right)_n=2\sum_{k=n+1}^{\infty}\sum_{l=k}^{\infty}\sqrt{\frac{b_{k-1}b_k}{(1+1/l)^\alpha}}T_l(1+\delta_l)\prod_{j=k}^{l}\frac{(1+1/j)^\alpha}{(1-1/j)^\alpha}.$$

From the choice of n_1 we have

$$\|\mathfrak{T}\left(\underline{0}\right)\|_{\ell^\infty\{n_1,\,n_1+1,\ldots\}}\le 1/3,\ \text{ and } \|D\mathfrak{T}\left(\underline{0}\right)\|_{\ell^\infty\{n_1,\,n_1+1,\ldots\}}\le 1/3.$$

Therefore by the contraction mapping principle, the equation

$$\underline{\delta}=\mathfrak{T}\left(\underline{\delta}\right)$$

has a unique solution $\underline{\tilde{\delta}}^1$ in $\ell^\infty\{n_1,\, n_1+1,\,\ldots\}$. This solution satisfies

$$\|\underline{\tilde{\delta}}^1\|_{\ell^\infty\{n_1,\,n_1+1,\ldots\}}\le\frac{1}{2}.$$

Using (8.18), and the estimate on T_n we get for any $n>n_1$

$$|\tilde{\delta}_{n+1}-\tilde{\delta}_n|\le\mathcal{O}(1)n^{-1-\zeta}.$$

We know from Proposition 22 that any solution of $Rw=w$ which behaves for large n like n^α has to be proportional to w_n^1. Therefore the result follows. □

Proposition 35. *There exists a constant $\tilde{C}>0$ such that for any $\epsilon\in]0,1]$ and for any*

$$1\le n\le N_1=\left[\frac{\tilde{C}}{\sqrt{\epsilon}}\right],\tag{8.19}$$

the equation $Rw=(1+\epsilon)w-\mathbf{1}_{n=1}$ has a unique (positive) solution (W_n) given by

$$W_n=w_n^1(1+\bar{\delta}_n^1)$$

with $\bar{\delta}_1^1=0$. This solution satisfies for any $1<n\le N_1$

$$|\bar{\delta}_n^1|\le\mathcal{O}(1)\cdot\epsilon\cdot n^2,$$

and for any $1<n<N_1$

$$|\bar{\delta}_{n+1}^1-\bar{\delta}_n^1|\le\mathcal{O}(1)\cdot\epsilon\cdot n.$$

Proof: We look for a solution w of equation (8.13) of the form

$$w_n=w_n^1(1+\delta_n).$$

Using this ansatz in equation (8.13) we get for $n>1$

$$\frac{w_{n+1}^1(1+\delta_{n+1})}{2\sqrt{b_nb_{n+1}}}+\frac{w_{n-1}^1(1+\delta_{n-1})}{2\sqrt{b_nb_{n-1}}}\mathbf{1}_{n>1}=w_n^1(1+\epsilon)(1+\delta_n).$$

This can be rearranged as

$$\frac{w_{n+1}^1\delta_{n+1}}{2\sqrt{b_nb_{n+1}}}+\frac{w_{n-1}^1\delta_{n-1}}{2\sqrt{b_nb_{n-1}}}\mathbf{1}_{n>1}-w_n^1\delta_n=\epsilon w_n^1(1+\delta_n).$$

For $n=1$ we get if $\delta_1=0$

$$\delta_2 = 2\epsilon\sqrt{b_2 b_1}\frac{w_1^1}{w_2^1}. \tag{8.20}$$

For $n>1$ we have

$$\frac{w_{n+1}^1(\delta_{n+1}-\delta_n)}{2\sqrt{b_n b_{n+1}}} - \frac{w_{n-1}^1(\delta_n-\delta_{n-1})}{2\sqrt{b_n b_{n-1}}} = \epsilon w_n^1(1+\delta_n).$$

hence

$$\delta_{n+1}-\delta_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{w_{n-1}^1}{w_{n+1}^1}(\delta_n-\delta_{n-1}) + 2\epsilon\sqrt{b_n b_{n+1}}\frac{w_n^1}{w_{n+1}^1}(1+\delta_n).$$

We deduce that for $n>1$ using equation (8.24) with $p=1$,

$$h_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{w_{n-1}^1}{w_{n+1}^1},\ g_n = 2\epsilon\sqrt{b_n b_{n+1}}\frac{w_n^1}{w_{n+1}^1}(1+\delta_n)$$

we get

$$\delta_{n+1}-\delta_n = \sqrt{\frac{b_{n+1}b_n}{b_1 b_2}}\frac{w_1^1 w_2^1}{w_{n+1}^1 w_n^1}(\delta_2-\delta_1) + 2\epsilon\frac{\sqrt{b_{n+1}b_n}}{w_n^1 w_{n+1}^1}\sum_{j=2}^{n}(w_j^1)^2(1+\delta_j).$$

Using equations (8.25) and (8.20), we get for $n>2$

$$\delta_n = \delta_2 + \sum_{l=2}^{n-1}(\delta_{l+1}-\delta_l)$$

$$= \delta_2 + \sum_{l=2}^{n-1}\sqrt{\frac{b_{l+1}b_l}{b_1 b_2}}\frac{w_1^1 w_2^1}{w_{l+1}^1 w_l^1}(\delta_2-\delta_1) + 2\epsilon\sum_{l=2}^{n-1}\frac{\sqrt{b_{l+1}b_l}}{w_l^1 w_{l+1}^1}\sum_{j=2}^{l}(w_j^1)^2(1+\delta_j)$$

$$= 2\epsilon\left(\sqrt{b_2 b_1}\frac{w_1^1}{w_2^1}(1+\delta_2)\sum_{l=1}^{n-1}\sqrt{\frac{b_{l+1}b_l}{b_1 b_2}}\frac{w_1^1 w_2^1}{w_{l+1}^1 w_l^1} + \sum_{l=2}^{n-1}\frac{\sqrt{b_{l+1}b_l}}{w_l^1 w_{l+1}^1}\sum_{j=2}^{l}(w_j^1)^2(1+\delta_j)\right).$$

We now consider the affine operator $\mathfrak{T}$ acting on $l^\infty(\{2,3,\ldots,N_1\})$ and defined by

$$\mathfrak{T}(\underline{\delta})_2 = 2\epsilon\sqrt{b_2 b_1}\frac{w_1^1}{w_2^1},$$

and for $2<n<N_1$

$$\mathfrak{T}(\underline{\delta})_n = 2\epsilon\left(\sqrt{b_2\,b_1}\frac{w_1^1}{w_2^1}(1+\delta_2)\sum_{l=2}^{n-1}\sqrt{\frac{b_{l+1}b_l}{b_1 b_2}}\frac{w_1^1 w_2^1}{w_{l+1}^1 w_l^1} + \sum_{l=2}^{n-1}\frac{\sqrt{b_{l+1}b_l}}{w_l^1 w_{l+1}^1}\sum_{j=2}^{l}(w_j^1)^2(1+\delta_j)\right).$$

We can now write the equation for δ^1

$$(I-\mathfrak{T})(\underline{\delta}^1)=0.$$

We know from Corollary 23 that there exists a constant $C_1>1$ such that for any $n\geq 1$

$$\frac{n^\alpha}{C_1}\leq w_n^1\leq C_1 n^\alpha.$$

Recall that $-1/2<\alpha<1/2$, and there exists a constant $C'>1$ such that for any $n\geq 1$

$$\frac{1}{C'}\leq b_n\leq C'.$$

We have

$$\|D\mathfrak{T}\,(\underline{0})\|_{l^\infty(\{2,3,\dots,N_1\})}$$
$$\le 2\,\epsilon\left(\sqrt{b_2 b_1}\frac{w_1^1}{w_2^1}\sum_{l=2}^{N_1-1}\sqrt{\frac{b_{l+1}b_l}{b_1 b_2}}\frac{w_1^1 w_2^1}{w_{l+1}^1 w_l^1}+\sum_{l=2}^{N_1-1}\frac{\sqrt{b_{l+1}b_l}}{w_l^1 w_{l+1}^1}\sum_{j=2}^{l}(w_j^1)^2\right)$$
$$=2\epsilon\left((w_1^1)^2\sum_{l=2}^{N_1-1}\sqrt{b_{l+1}b_l}\frac{1}{w_{l+1}^1 w_l^1}+\sum_{l=2}^{N_1-1}\frac{\sqrt{b_{l+1}b_l}}{w_l^1 w_{l+1}^1}\sum_{j=2}^{l}(w_j^1)^2\right)$$
$$=2\epsilon\sum_{l=2}^{N_1-1}\frac{\sqrt{b_{l+1}b_l}}{w_l^1 w_{l+1}^1}\sum_{j=1}^{l}(w_j^1)^2\le 2\,\epsilon C' C_1^4\sum_{l=2}^{N_1-1}\frac{1}{l^\alpha(l+1)^\alpha}\sum_{j=1}^{l}j^{2\alpha}\le 2\epsilon C' C_1^4 C_\alpha^{''} N_1^2$$

since $-1/2<\alpha<1/2$, where $C_\alpha^{''}>0$ is chosen as

$$C_\alpha^{''}=\max\left\{\sup_{n\ge 3} n^{-2}\sum_{l=2}^{n-1}\frac{1}{l^\alpha\,(l+1)^\alpha}\sum_{j=1}^{l}j^{2\alpha},\ 1\right\}<\infty.$$

Therefore, if N_1 is such that

$$2\,\epsilon C' C_1^4 C_\alpha^{''} N_1^2<\frac{1}{3},$$

the linear map $I-D\mathfrak{T}$ is invertible with an inverse of norm at most $3/2$. In this case, we define

$$\underline{\bar{\delta}}^1=\left(I-D\mathfrak{T}\,(\underline{0})\right)^{-1}\mathfrak{T}\,(\underline{0})\,,$$

We have also

$$\|\mathfrak{T}\,(\underline{0})\|_{l^\infty(\{2,3,\dots,N_1\})}\le 2\,\epsilon C' C_1^4 C_\alpha^{''} N_1^2<\frac{1}{3},$$

This implies

$$\|\underline{\bar{\delta}}^1\|_{l^\infty(\{2,3,\dots,N_1\})}\le\frac{1}{2}.$$

From equation (8.25) and the above estimates it follows that for $1\le n\le N_1$

$$\left|\bar{\delta}_n^1\right|\le\mathcal{O}(1)\cdot\epsilon\cdot n^2,$$

and from equation (8.24) we have for $2\le n\le N_1-1$

$$\left|\bar{\delta}_{n+1}^1-\bar{\delta}_n^1\right|\le\mathcal{O}(1)\cdot\epsilon\cdot n^{-2\alpha}+\mathcal{O}(1)\cdot\epsilon\cdot n\le\mathcal{O}(1)\cdot\epsilon\cdot n.\ \square$$

Proposition 36. *There exists a constant $\tilde{C}>0$, an integer $n_1>1$, and a constant $0<\epsilon_0\le 1$ such that for any $\epsilon\in]0,\epsilon_0]$ and for any*

$$n_1\le n\le N_1=\left[\frac{\tilde{C}}{\sqrt{\epsilon}}\right],$$

the solution of equation $Rw=(1+\epsilon)w-\mathbf{1}_{n=1}$ constructed in Proposition 34 satisfies

$$W_n=\tilde{C}n^\alpha(1+\delta_n^1).$$

with

$$\sup_{n_1\le n<N_1}|\delta_n^1|\le\mathcal{O}(1)$$

and for any $n_1\le n<N_1$

$$|\delta_{n+1}^1-\delta_n^1|\le\mathcal{O}(1)\cdot\left(\epsilon\cdot n+n^{-1-\zeta}\right).$$

Proof: We take $\epsilon_0 > 0$ small enough such that

$$\left[\frac{\tilde{C}}{\sqrt{\epsilon_0}}\right] > n_1 + 3$$

where n_1 is given in Proposition 34. The Proposition follows by combining the results of propositions 34 and 35. $\Box$

- Zone 2, $n \approx N(\epsilon)$.

First the idea. We look for a function f such that $f(n\sqrt{\epsilon})$ is almost a solution of $(Rw)_n = (1+\epsilon)w_n$ for $n \approx N$. We have

$$\frac{1}{2\sqrt{b_n b_{n+1}}} f((n+1)\sqrt{\epsilon}) + \frac{1}{2\sqrt{b_n b_{n-1}}} f((n-1)\sqrt{\epsilon}) - (1+\epsilon) f(n\sqrt{\epsilon}) =$$

$$f(n\sqrt{\epsilon})\left(\frac{1}{2\sqrt{b_n b_{n+1}}} + \frac{1}{2\sqrt{b_n b_{n-1}}} - 1 - \epsilon\right) + \sqrt{\epsilon} f'(n\sqrt{\epsilon})\left(\frac{1}{2\sqrt{b_n b_{n+1}}} - \frac{1}{2\sqrt{b_n b_{n-1}}}\right)$$

$$+\frac{\epsilon}{2} f''(n\sqrt{\epsilon})\left(\frac{1}{2\sqrt{b_n b_{n+1}}} + \frac{1}{2\sqrt{b_n b_{n-1}}}\right) + \mathcal{O}(\epsilon^{3/2}) \cdot f^{'''}(\xi_{n+1}) + \mathcal{O}(\epsilon^{3/2}) \cdot f^{'''}(\xi_{n-1})$$

for some $\xi_{n+1} \in [n\sqrt{\epsilon}, (n+1)\sqrt{\epsilon}]$ and $\xi_{n-1} \in [(n-1)\sqrt{\epsilon}, n\sqrt{\epsilon}]$. This is also equal to

$$\frac{\epsilon}{2} f''(n\sqrt{\epsilon}) + \epsilon f(n\sqrt{\epsilon})\left(-1 + \frac{\mathrm{w}}{\epsilon \cdot n^2}\right)$$
$$+\mathcal{O}(1) \cdot n^{-2-\zeta} f(n\sqrt{\epsilon}) + \mathcal{O}(1) \cdot \sqrt{\epsilon} n^{-2-\zeta} f'(n\sqrt{\epsilon}) + \mathcal{O}(1) \cdot \epsilon \cdot n^{-2} f''(n\sqrt{\epsilon})$$
$$+\mathcal{O}(\epsilon^{3/2}) \cdot f^{'''}(\xi_{n+1}) + \mathcal{O}(\epsilon^{3/2}) \cdot f^{'''}(\xi_{n-1}).$$

We now choose f as in (8.14) and look for an exact solution of the equation $(Rw)_n = (1+\epsilon)\, w_n$ for $n \geq N$, of the form

$$w_n = f(n\sqrt{\epsilon})(1+\delta_n),$$

with the sequence (δ_n) small (for $n \geq N$). We get

$$\frac{f((n+1)\sqrt{\epsilon})(1+\delta_{n+1})}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})(1+\delta_{n-1})}{2\sqrt{b_n b_{n-1}}} - (1+\epsilon) f(n\sqrt{\epsilon})(1+\delta_n) =$$

$$\frac{f((n+1)\sqrt{\epsilon})\delta_{n+1}}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})\delta_{n-1}}{2\sqrt{b_n b_{n-1}}} - (1+\epsilon) f(n\sqrt{\epsilon})\delta_n + R_n =$$

$$\frac{f((n+1)\sqrt{\epsilon})\delta_{n+1}}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})\delta_{n-1}}{2\sqrt{b_n b_{n-1}}} - \left(\frac{f((n+1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n-1}}}\right)\delta_n$$
$$+(1+\delta_n)R_n,$$

where

$$R_n = \frac{f((n+1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n-1}}} - (1+\epsilon) f(n\sqrt{\epsilon}).$$

Since $\zeta \leq 1$, we have

$$|R_n| \leq \mathcal{O}(1) \cdot \left(\frac{f(n\sqrt{\epsilon})}{n^{2+\zeta}} + \epsilon^{3/2} \sup_{\xi \in [(n-1)\sqrt{\epsilon}, (n+1)\sqrt{\epsilon})]} |f'''(\xi)|\right).$$

Therefore, after some easy algebra using $0 < \zeta \le 1$ we get

$$|R_n| \le \mathcal{O}(1) \cdot \begin{cases} n^{\alpha-2-\zeta} & \text{if } n \le N \\ \epsilon^{1+\zeta/2-\alpha/2}\ e^{-\,n\sqrt{2\epsilon}} & \text{if } n \ge N \end{cases}$$

We now consider the equation for (δ_n)

$$\frac{f((n+1)\sqrt{\epsilon})\delta_{n+1}}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})\delta_{n-1}}{2\sqrt{b_n b_{n-1}}} - \left(\frac{f((n+1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n+1}}} + \frac{f((n-1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n-1}}}\right)\delta_n = -(1+\delta_n)R_n.$$

We can rewrite this as

$$\frac{f((n+1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n+1}}}(\delta_{n+1}-\delta_n) - \frac{f((n-1)\sqrt{\epsilon})}{2\sqrt{b_n b_{n-1}}}(\delta_n-\delta_{n-1}) = -(1+\delta_n)R_n.$$

This can be rewritten

$$\delta_{n+1}-\delta_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{f((n-1)\sqrt{\epsilon})}{f((n+1)\sqrt{\epsilon})}(\delta_n-\delta_{n-1}) - 2\frac{\sqrt{b_n b_{n+1}}}{f((n+1)\sqrt{\epsilon})}(1+\delta_n)R_n.$$

• Case $n \ge N$.

For $n \ge N$ we take $\delta_N = 0$ and $\delta_{N+1} = 0$. We apply equation (8.25) with $p = N$,

$$h_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}}\frac{f((n-1)\sqrt{\epsilon})}{f((n+1)\sqrt{\epsilon})},\ g_n = -2\frac{\sqrt{b_n b_{n+1}}}{f((n+1)\sqrt{\epsilon})}(1+\delta_n)R_n.$$

We get for $n \ge N+2$

$$\delta_n = -2\sum_{k=N+1}^{n-1}\frac{\sqrt{b_{k+1}b_k}}{f((k+1)\sqrt{\epsilon})}(1+\delta_k)R_k\sum_{l=k}^{n-1}\sqrt{\frac{b_{l+1}b_l}{b_{k+1}b_k}}\frac{f((k+1)\sqrt{\epsilon})f(k\sqrt{\epsilon})}{f((l+1)\sqrt{\epsilon})f((l)\sqrt{\epsilon})}$$

Using the estimates on R_n, f and b, it is easy to prove that there exists a constant $C'' > 0$ such that for any $\epsilon \in]0,1]$, and any $n \ge N+2$

$$\sum_{k=N+1}^{n-1}\frac{\sqrt{b_{k+1}b_k}}{f((k+1)\sqrt{\epsilon})}|R_k|\sum_{l=k}^{n-1}\sqrt{\frac{b_{l+1}b_l}{b_{k+1}b_k}}\frac{f((k+1)\sqrt{\epsilon})f(k\sqrt{\epsilon})}{f((l+1)\sqrt{\epsilon})f(l\sqrt{\epsilon})}$$

$$\le \mathcal{O}(1)\cdot\sum_{k=N+1}^{n-1}\epsilon^{1+\zeta/2}\sum_{l=k}^{n-1}e^{2(l-k)\sqrt{2\epsilon}} \le \mathcal{O}(1)\cdot\epsilon^{1/2+\zeta/2}\sum_{k=N+1}^{n-1}e^{2(n-k)\sqrt{2\epsilon}}$$

$$\le C''\epsilon^{\zeta/2}e^{2(n-N)\sqrt{2\epsilon}}$$

We define an affine operator $\mathfrak{T}$ on $l^\infty(\{N, N+1, \dots M'\})$ by

(8.21)

$$\mathfrak{T}(\underline{\delta})_n = -2\sum_{k=N+1}^{n-1}\frac{\sqrt{b_{k+1}b_k}}{f((k+1)\sqrt{\epsilon})}(1+\delta_k)R_k\sum_{l=k}^{n-1}\sqrt{\frac{b_{l+1}b_l}{b_{k+1}b_k}}\frac{f((k+1)\sqrt{\epsilon})f(k\sqrt{\epsilon})}{f((l+1)\sqrt{\epsilon})f(l\sqrt{\epsilon})},$$

for $N+2 \le n \le M'$, and

$$\mathfrak{T}(\underline{\delta})_N = \mathfrak{T}(\underline{\delta})_{N+1} = 0$$

We have

$$\|D\mathfrak{T}(\underline{0})\|_{l^\infty(\{N,N+1,\dots M'\})} \le C''\epsilon^{\zeta/2}e^{2(M'-N)\sqrt{2\epsilon}}$$

and

$$\|\mathfrak{T}(\underline{0})\|_{l^\infty(\{N,N+1,\dots M'\})} \le C''\epsilon^{\zeta/2}e^{2(M'-N)\sqrt{2\,\epsilon}}.$$

For $0<\epsilon<1$, we choose

$$M' = N + \left[-\frac{\log\left(3C''\epsilon^{\zeta/2}\right)}{\sqrt{2\epsilon}} \right],$$

then

$$C''\epsilon^{\zeta/2}e^{2(M'-N)\sqrt{2\epsilon}} < \frac{1}{3}.$$

Note that for ϵ small we have $M' \gg N$. The equation

$$\underline{\delta} = \mathfrak{T}\left(\underline{\delta}\right)$$

has a unique solution $\underline{\delta}^2$ in $l^\infty\left(\{N, N+1, \dots M'\}\right)$, and

$$\left\|\underline{\delta}^2\right\|_{l^\infty(\{N,N+1,\dots M'\})} \le \frac{1}{2}.$$

From equation (8.24) we have

$$\delta_{n+1} - \delta_n = \sum_{k=N+1}^{n} g_k \prod_{j=k+1}^{n} h_j$$

$$= -2\sum_{k=N+1}^{n} \frac{\sqrt{b_k b_{k+1}}}{f((k+1)\sqrt{\epsilon})}(1+\delta_k)R_k \prod_{j=k+1}^{n} \sqrt{\frac{b_{j+1}}{b_{j-1}}\frac{f((j-1)\sqrt{\epsilon})}{f((j+1)\sqrt{\epsilon})}}$$

$$= -2\sum_{k=N+1}^{n} \frac{\sqrt{b_k b_{k+1}}}{f((k+1)\sqrt{\epsilon})}(1+\delta_k)R_k \sqrt{\frac{b_{n+1}b_n}{b_{k+1}b_k}\frac{f((k+1)\sqrt{\epsilon})f(k\sqrt{\epsilon})}{f((n+1)\sqrt{\epsilon})f(n\sqrt{\epsilon})}}.$$

Therefore, if $N \le n \le M'$

$$|\delta_{n+1} - \delta_n| \le C''\epsilon^{1/2+\zeta/2}e^{2(n-N)\sqrt{2\epsilon}} \tag{8.22}$$

• Case $n \le N+1$.

For $n \le N+1$ we take $\delta_N = 0$ and $\delta_{N+1} = 0$ as before to obtain the same solution. We apply equation (8.27) with $p = N+1$

$$h_n = \sqrt{\frac{b_{n+1}}{b_{n-1}}\frac{f((n-1)\sqrt{\epsilon})}{f((n+1)\sqrt{\epsilon})}},\ g_n = -2\frac{\sqrt{b_n b_{n+1}}}{f((n+1)\sqrt{\epsilon})}(1+\delta_n)R_n.$$

We get for $n<N$

$$\delta_n = -2\sum_{l=n+1}^{N} \frac{\sqrt{b_l b_{l+1}}}{f((l+1)\sqrt{\epsilon})}(1+\delta_l)R_l \sum_{k=n+1}^{l} \sqrt{\frac{b_k b_{k-1}}{b_{l+1}b_l}}\frac{f(l\sqrt{\epsilon})f((l+1)\sqrt{\epsilon})}{f((k-1)\sqrt{\epsilon})\,f(k\sqrt{\epsilon})}$$

Using the estimates on R_n, f and b, it is easy to prove that there exists a constant $C''' > 0$ such that for any $\epsilon \in]0,1]$, and any $n<N$

$$\sum_{l=n+1}^{N} \frac{\sqrt{b_l b_{l+1}}}{f((l+1)\sqrt{\epsilon})}|R_l| \sum_{k=n+1}^{l} \sqrt{\frac{b_k b_{k-1}}{b_{l+1}b_l}}\frac{f(l\sqrt{\epsilon})f((l+1)\sqrt{\epsilon})}{f((k-1)\sqrt{\epsilon})f(k\sqrt{\epsilon})}$$

$$\le \mathcal{O}(1)\cdot\sum_{l=n+1}^{N} l^{-2-\zeta}\sum_{k=n+1}^{l}\frac{l^{2\alpha}}{k^{2\alpha}} \le \mathcal{O}(1)\cdot\frac{1}{n^{2\alpha-1}}\sum_{l=n+1}^{N} l^{-2-\zeta+2\alpha} \le C'''n^{-\zeta}$$

since $-1/2<\alpha<1/2$.

We define an affine operator $\mathfrak{T}$ on $l^\infty(\{M, M+1, \dots N+1\})$ by

$$\mathfrak{T}(\underline{\delta})_n = -2\sum_{l=n+1}^{N}\frac{\sqrt{b_l b_{l+1}}}{f((l+1)\sqrt{\epsilon})}(1+\delta_l)R_l\sum_{k=n+1}^{l}\sqrt{\frac{b_k b_{k-1}}{b_{l+1}b_l}}\frac{f(l\sqrt{\epsilon})f((l+1)\sqrt{\epsilon})}{f((k-1)\sqrt{\epsilon})f(k\sqrt{\epsilon})},$$

for $M \le n \le N-1$, and

$$\mathfrak{T}(\underline{\delta})_N = \mathfrak{T}(\underline{\delta})_{N+1} = 0.$$

We have

$$\|D\mathfrak{T}(\underline{0})\|_{l^\infty(\{M,M+1,\dots N+1\})} \le C''' M^{-\zeta},$$

and

$$\|\mathfrak{T}(\underline{0})\|_{l^\infty(\{M,M+1,\dots N+1\})} \le C''' M^{-\zeta}.$$

Hence for $M > 0$ large enough, namely

$$C''' M^{-\zeta} < \frac{1}{2},$$

the equation

$$\underline{\delta} = \mathfrak{T}(\underline{\delta})$$

has a unique solution δ^2 in $l^\infty(\{M, M+1, \dots N+1\})$, and

$$\|\underline{\delta^2}\|_{l^\infty(\{M,M+1,\dots N+1\})} \le \frac{1}{2}.$$

Using equation (8.26), we get for any $M \le n < N$

$$\left|\delta^2_{n+1} - \delta^2_n\right| \le \mathcal{O}(1)\cdot n^{-1-\zeta}.$$

We therefore obtain:

Proposition 37. *There exist three positive constants $\bar{C}$, $\bar{C}'$ and $\epsilon_0 \in]0,1[$ such that for any $\epsilon \in]0,\epsilon_0[$ and for any $n \in [M, M']$ with*

$$M = \bar{C},\ \ M' = \left[\bar{C}'\epsilon^{-1/2}\log\epsilon^{-1}\right], \tag{8.23}$$

the equation $Rw = (1+\epsilon)w$ has a positive solution (F_n) of the form ($\nu = 1/2-\alpha$)

$$F_n = f\left(n\sqrt{\epsilon}\right)(1+\delta^2_n) = \epsilon^{1/4-\alpha/2}\sqrt{n}K_\nu\left(n\sqrt{2\epsilon}\right)(1+\delta^2_n)$$

with $\delta^2_N = \delta^2_{N+1} = 0$ and $|\delta^2_n| \le 1/2$ and

$$|\delta^2_{n+1} - \delta^2_n| \le \begin{cases}\mathcal{O}(1)\ n^{-1-\zeta} & \text{for } n \le N\ ,\\ \mathcal{O}(1)\ \epsilon^{1/2+\zeta/2}\, e^{2\,(n-N)\,\sqrt{2\,\epsilon}} & \text{for } n \ge N\ .\end{cases}$$

Proof: Match the two latter pieces obtained while $n \ge N$ and $n \le N+1$, at N and $N+1$. □

$A.15$ PROPAGATORS AND WRONSKIANS.

In this last Appendix, we give some supplementary material needed in particular in Appendix $A.14$.

- **Propagators.** For $n \ge p$ assume

$$\delta_{n+1} - \delta_n = h_n(\delta_n - \delta_{n-1}) + g_n$$

We define $R_{p+1} = \delta_{p+1} - \delta_p$, and for $n > p$

$$\delta_{n+1} - \delta_n = R_{n+1} \prod_{j=p+1}^{n} h_j .$$

Then

$$R_{n+1} \prod_{j=p+1}^{n} h_j = h_n R_n \prod_{j=p+1}^{n-1} h_j + g_n$$

$$R_{n+1} = R_n + g_n \prod_{j=p+1}^{n} \frac{1}{h_j}$$

$$R_{n+1} = R_{p+1} + \sum_{k=p+1}^{n} (R_{k+1} - R_k)$$

$$R_{n+1} = R_{p+1} + \sum_{k=p+1}^{n} g_k \prod_{j=p+1}^{k} \frac{1}{h_j}$$

$$\delta_{n+1} - \delta_n = (\delta_{p+1} - \delta_p) \prod_{j=p+1}^{n} h_j + \prod_{j=p+1}^{n} h_j \sum_{k=p+1}^{n} g_k \prod_{j=p+1}^{k} \frac{1}{h_j}$$

$$= (\delta_{p+1} - \delta_p) \prod_{j=p+1}^{n} h_j + \sum_{k=p+1}^{n} g_k \prod_{j=k+1}^{n} h_j . \tag{8.24}$$

For $n > p + 1$,

$$\delta_n = \delta_{p+1} + \sum_{l=p+1}^{n-1} (\delta_{l+1} - \delta_l)$$

$$= \delta_{p+1} + (\delta_{p+1} - \delta_p) \sum_{l=p+1}^{n-1} \prod_{j=p+1}^{l} h_j + \sum_{l=p+1}^{n-1} \sum_{k=p+1}^{l} g_k \prod_{j=k+1}^{l} h_j$$

$$= \delta_p + (\delta_{p+1} - \delta_p) \left(1 + \sum_{l=p+1}^{n-1} \prod_{j=p+1}^{l} h_j \right) + \sum_{l=p+1}^{n-1} \sum_{k=p+1}^{l} g_k \prod_{j=k+1}^{l} h_j$$

$$= \delta_p + (\delta_{p+1} - \delta_p) \left(1 + \sum_{l=p+1}^{n-1} \prod_{j=p+1}^{l} h_j \right) + \sum_{k=p+1}^{n-1} g_k \sum_{l=k}^{n-1} \prod_{j=k+1}^{l} h_j . \tag{8.25}$$

For $1 < n < p$ assume

$$\delta_{n-1} - \delta_n = \frac{1}{h_n} (\delta_n - \delta_{n+1}) + \frac{g_n}{h_n}$$

We define $R_{p-1} = \delta_{p-1} - \delta_p$, and for $n < p$

$$\delta_{n-1} - \delta_n = R_{n-1} \prod_{j=n}^{p-1} \frac{1}{h_j} .$$

Then

$$R_{n-1} \prod_{j=n}^{p-1} \frac{1}{h_j} = \frac{1}{h_n} R_n \prod_{j=n+1}^{p-1} \frac{1}{h_j} + \frac{g_n}{h_n}$$

$$R_{n-1} = R_n + g_n \prod_{j=n+1}^{p-1} h_j$$

$$\begin{aligned} R_{n-1} &= R_{p-1} + \sum_{l=n}^{p-1} (R_{l-1} - R_l) \\ &= R_{p-1} + \sum_{l=n}^{p-1} g_l \prod_{j=l+1}^{p-1} h_j. \end{aligned}$$

For $k < p$

$$\begin{aligned} \delta_{k-1} - \delta_k &= (\delta_{p-1} - \delta_p) \prod_{j=k}^{p-1} \frac{1}{h_j} + \prod_{j=k}^{p-1} \frac{1}{h_j} \sum_{l=k}^{p-1} g_l \prod_{j=l+1}^{p-1} h_j \\ &= (\delta_{p-1} - \delta_p) \prod_{j=k}^{p-1} \frac{1}{h_j} + \sum_{l=k}^{p-1} g_l \prod_{j=k}^{l} \frac{1}{h_j} \end{aligned} \tag{8.26}$$

and for $n < p-1$

$$\begin{aligned} \delta_n &= \delta_{p-1} + \sum_{k=n+1}^{p-1} (\delta_{k-1} - \delta_k) \\ &= \delta_{p-1} + (\delta_{p-1} - \delta_p) \sum_{k=n+1}^{p-1} \prod_{j=k}^{p-1} \frac{1}{h_j} + \sum_{k=n+1}^{p-1} \sum_{l=k}^{p-1} g_l \prod_{j=k}^{l} \frac{1}{h_j} \\ &= \delta_p + (\delta_{p-1} - \delta_p) \left(1 + \sum_{k=n+1}^{p-1} \prod_{j=k}^{p-1} \frac{1}{h_j} \right) + \sum_{k=n+1}^{p-1} \sum_{l=k}^{p-1} g_l \prod_{j=k}^{l} \frac{1}{h_j}. \\ &= \delta_p + (\delta_{p-1} - \delta_p) \left(1 + \sum_{k=n+1}^{p-1} \prod_{j=k}^{p-1} \frac{1}{h_j} \right) + \sum_{l=n+1}^{p-1} g_l \sum_{k=n+1}^{l} \prod_{j=k}^{l} \frac{1}{h_j}. \end{aligned} \tag{8.27}$$

• **Cancellation of Wronskians.** Let

$$X_n = x_n(1 + \delta_n^x), \ Y_n = y_n(1 + \delta_n^y).$$

Then

$$\begin{aligned} W(X,Y)_n &= W(x,y)_n \left(1 + \delta_n^x + \delta_{n+1}^y + \delta_n^x \delta_{n+1}^y\right) + \\ &+ y_n x_{n+1} \left((\delta_{n+1}^x - \delta_n^x)(1 + \delta_n^y) - (\delta_{n+1}^y - \delta_n^y)(1 + \delta_n^x) \right). \end{aligned} \tag{8.28}$$

Another version of this fact is as follows. Let

$$X_n = x_n u_n, \ Y_n = y_n v_n.$$

Then

$$W(X,Y)_n = W(x,y)_n u_n v_{n+1} + y_n x_{n+1} \left((u_{n+1} - u_n) v_n - (v_{n+1} - v_n) u_n \right).$$

• **Other solutions and Wronskians.** Let (x_n) and (y_n) satisfy

$$\frac{x_{n+1}}{2\sqrt{b_n b_{n+1}}}+\frac{x_{n-1}}{2\sqrt{b_n b_{n-1}}}=\rho x_n \text{ and } \frac{y_{n+1}}{2\sqrt{b_n b_{n+1}}}+\frac{y_{n-1}}{2\sqrt{b_n b_{n-1}}}=\rho y_n.$$

for $M\le n\le M'$. Let $p\in]M,M'[$, and assume y_p and y_{p+1} are given. We have

$$W(x,y)_n=\sqrt{\frac{b_{n+1}b_n}{b_n b_{n-1}}}W(x,y)_{n-1}$$

and (see Lemma 14)

$$W(x,y)_n=\sqrt{\frac{b_{n+1}b_n}{b_{p+1}b_p}}W(x,y)_p\,.$$

Then

$$\frac{y_{n+1}}{x_{n+1}}=\frac{y_n}{x_n}+\frac{W(y,x)_n}{x_n x_{n+1}}$$

and for $n>p$

$$\frac{y_n}{x_n}=\sum_{l=p}^{n-1}\left(\frac{y_{l+1}}{x_{l+1}}-\frac{y_l}{x_l}\right)+\frac{y_p}{x_p}=\sum_{l=p}^{n-1}\frac{W(y,x)_l}{x_l x_{l+1}}+\frac{y_p}{x_p}$$

$$=\frac{y_p}{x_p}+\frac{W(y,x)_p}{\sqrt{b_{p+1}b_p}}\sum_{l=p}^{n-1}\frac{\sqrt{b_{l+1}b_l}}{x_l x_{l+1}} \tag{8.29}$$

hence

$$y_n=\frac{y_p}{x_p}x_n+x_n\frac{W(y,x)_p}{\sqrt{b_{p+1}b_p}}\sum_{l=p}^{n-1}\frac{\sqrt{b_{l+1}b_l}}{x_l x_{l+1}}.$$

For $n<p$

$$\frac{y_n}{x_n}=-\sum_{l=n}^{p-1}\left(\frac{y_{l+1}}{x_{l+1}}-\frac{y_l}{x_l}\right)+\frac{y_p}{x_p}=-\sum_{l=n}^{p-1}\frac{W(y,x)_l}{x_l x_{l+1}}+\frac{y_p}{x_p}$$

$$=\frac{y_p}{x_p}-\frac{W(y,x)_p}{\sqrt{b_{p+1}b_p}}\sum_{l=n}^{p-1}\frac{\sqrt{b_{l+1}b_l}}{x_l x_{l+1}} \tag{8.30}$$

hence

$$y_n=\frac{y_p}{x_p}x_n-x_n\frac{W(x,y)_p}{\sqrt{b_{p+1}b_p}}\sum_{l=n}^{p-1}\frac{\sqrt{b_{l+1}b_l}}{x_l x_{l+1}}.$$

If we define a sequence $(\tilde{y}_n)$ by

$$\tilde{y}_n=\begin{cases}-\sum_{l=n}^{p-1}\frac{\sqrt{b_{\ell+1}\,b_\ell}}{x_\ell\,x_{\ell+1}} & \text{if } n<p\\ 0 & \text{if } n=p\\ \sum_{l=p}^{n-1}\frac{\sqrt{b_{\ell+1}\,b_\ell}}{x_\ell\,x_{\ell+1}} & \text{if } n>p\end{cases}$$

we have in all cases

$$y_n=\frac{y_p}{x_p}x_n+x_n\frac{W(y,x)_p}{\sqrt{b_{p+1}b_p}}\tilde{y}_n.$$

For two sequences (x_n) and (u_n) denote by xu the Hadamard product sequence

$$(xu)_n=x_n u_n.$$

Then

$$W\left(z,xu\right)_n = z_{n+1}x_nu_n - z_nx_{n+1}u_{n+1} = (z_{n+1}x_n - z_nx_{n+1})u_n - z_nx_{n+1}(u_{n+1}-u_n)$$
$$= W\left(z,x\right)_n u_{n+1} - z_nx_{n+1}(u_{n+1}-u_n).$$

In particular with $y = xu$ and

$$u_n = \frac{y_p}{x_p} + \frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\tilde{y}_n$$

$$W\left(z,y\right)_n = W\left(z,x\right)_n\left(\frac{y_p}{x_p} + \frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\tilde{y}_{n+1}\right) - z_nx_{n+1}\frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\left(\tilde{y}_{n+1}-\tilde{y}_n\right)$$
$$= W\left(z,x\right)_n\left(\frac{y_p}{x_p} + \frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\tilde{y}_{n+1}\right) - z_nx_{n+1}\frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\frac{\sqrt{b_{n+1}b_n}}{x_nx_{n+1}},$$

and finally

$$W\left(z,y\right)_n = W\left(z,x\right)_n\left(\frac{y_p}{x_p} + \frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\tilde{y}_{n+1}\right) - z_n\frac{W\left(y,x\right)_p}{\sqrt{b_{p+1}b_p}}\frac{\sqrt{b_{n+1}b_n}}{x_n}. \tag{8.31}$$

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